Computability Final Test — 2013-01-18

Notes.

- Answer both theory questions, and choose and solve two exercises, only. Solving more exercises results in the failure of the test.
- To pass the exam you need to provide a reasonable contribution in *both* Theory and Exercises parts.
- To achieve a score ≥ 28 you have to solve the exercise marked with $\star$ below.
- Significantly wrong answers will result in negative scores.
- Always provide a justification for your answers.

Reminder: when equating the results of partial functions (as in $\phi_i(3) = \phi_i(5)$), we mean that either 1) both sides of the equation are defined, and evaluate to the same natural number, or 2) both sides are undefined.

Theory

Question 1. *Prove that $\mathcal{RE}$ sets are closed under binary intersection and union, but not under complement.*

Question 2. *State and prove the Rice-Shapiro theorem (part $\Rightarrow$, only).*

Exercises

Exercise 3. *Prove whether*

$$A = \{n \mid \forall x \in \mathbb{N}.\ \phi_n(x^2) = 22\} \in \mathcal{RE}$$

Solution (sketch). $A \notin \mathcal{RE}$, by Rice-Shapiro ($\Rightarrow$). A is clearly semantically closed (easy to check), so let $\mathcal{F}_A$ denote the associated set of functions.

Take the constant function $f(n) = 22$. We have $f \in \mathcal{F}_A$ since $f(x^2) = 22$ for all x.

However no finite restriction g of f can belong to $\mathcal{F}_A$ since in that case g would be defined on $1^2, 2^2, 3^2, 4^2, \ldots$ hence on infinitely many points. □

Exercise 4. *Let $A = \{n \mid \phi_n(3) = 3\}$ and $B = \{n \mid \phi_n(5) = 5\}$. Prove whether $f(n) = \chi_A(n) + \chi_B(n)$ is a recursive total function.*

Solution (sketch). f is total (sum of two total functions), but not recursive. By contradiction, assume it is recursive. Hence, the following function g is recursive:

$$g(n) = \begin{cases} 1 & \text{if } f(n) = 2 \\ 0 & \text{otherwise} \end{cases}$$

(g is recursive since f is recursive and total, so $f(n) = 2$ is a recursive property).

We now prove that $\chi_{A\cap B}(n) = g(n)$. Indeed if $n \in A \cap B$ then $f(n) = 1+1 = 2$, so $g(n) = 1$. Otherwise, if $n \notin A \cap B$ then $f(n) = 0$ or 1, so $g(n) = 0$.

Hence, $A \cap B$ is recursive. However, $A \cap B = \{n \mid \phi_n(3) = 3 \wedge \phi_n(5) = 5\}$ which is not recursive by Rice (easy to check). This is a contradiction. □

Exercise 5. *Prove whether*

$$A = \{n \,|\, n \neq 0 \wedge \forall a, b \in \mathbb{N}.(a \cdot b = n \implies \phi_n(a) = 0)\} \in \mathcal{RE}$$

Solution (sketch). $A \in \mathcal{RE}$ since the following is a semi-verifier:

```
procedure S_A(n):
if n=0 then loop forever
for a := 1 to n do
   if n mod a = 0 then
      run φ_n(a) and take its result r
      if r ≠ 0 then loop forever
return 1
```

Clearly the code above works as intended for $n = 0$. In the other cases:

If $n \in A$, all the divisors a of n satisfy $\phi_n(a) = 0$ hence all the calls to the self-interpreter above halt and return $z = 0$. Hence, the `for` loop completes and the last line returns 1.

Otherwise, if $n \notin A$, we have $\phi_n(a) \neq 0$ for some divisor a of n. Let a be the minimum such divisor. If $\phi_n(a) = undefined$, then the `for` loop will get stuck by invoking the self-interpreter. If $\phi_n(a)$ is defined to some natural $\neq 0$, then `for` loop will get stuck in the explicit infinite loop after the check for $z = 0$. In either case, $S_A(n)$ loops forever, as it should. □

Exercise 6. ⋆ *Two sets $A, B \subseteq \mathbb{N}$ are said to be* separable *iff*

$$\exists f \in \mathcal{R}.\ (\forall a \in A.\ f(a) = 1) \wedge (\forall b \in B.\ f(b) = 0)$$

- *[5% score] State the* T, U*-normal form.*
- *[95% score] Prove whether* $A = \{\mathsf{pair}(i, j) \mid \phi_i, \phi_j \in \mathcal{PR} \wedge \phi_i = \phi_j\}$ *and* $B = \{\mathsf{pair}(i, j) \mid \phi_i, \phi_j \in \mathcal{PR} \wedge \phi_i \neq \phi_j\}$ *are separable.*

Solution (sketch). Intentionally omitted. □