Computability Assignment
Year 2012/13 - Number 11

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Note

Remember that $undefined \not> x$ for any natural x.

1 Question

Consider the set

$$A = \{n \mid \forall x \in \mathbb{N}.\ \phi_n(x) > x\}$$

Prove that $\mathsf{K} \leq_m A$.

1.1 Answer

We write

$$h(n) = \#\left(\lambda x.|\phi_n(n)| + x + 1\right)$$

We see that $h \in \mathcal{R}$ total (RZ: ok, in an exam add more justification). Then we show it is a reduction from K to A.

- if $n \in \mathsf{K}$, $\phi_n(n)$ is defined, $|\phi_n(n)| \geqslant 0$, thus $\forall x.\phi_{h(n)}(x) \geqslant x + 1 > x$, i.e. $h(n) \in A$
- if $n \notin \mathsf{K}$, $\phi_n(n)$ is undefined, $\phi_{h(n)}(x)$ is also undefined for any x, thus $h(n) \notin A$.

Thus $\mathsf{K} \leqslant_m A$.

2 Question

Prove that $\bar{\mathsf{K}} \leq_m A$, with the above A.

2.1 Answer

We write

$$h(n) = \#\left(\lambda x. \begin{cases} undefined & \text{if } \phi_n(n) \text{stops within } x \text{steps} \\ x+1 & o.w. \end{cases}\right)$$

Such h is in $\mathcal{R}$ total. We show h reduces $\bar{\mathsf{K}}$ to A.

- if $n \in \bar{\mathsf{K}}$, $\phi_n(n)$ is undefined, thus it never stops, $\forall x.\phi_{h(n)}(x) = x+1 > x$, i.e. $h(n) \in A$
- if $n \notin \bar{\mathsf{K}}$, $\phi_n(n)$ halts in some k steps,

  $$\phi_{h(n)}(x) = \begin{cases} undefined & k \leqslant x \\ x+1 & o.w. \end{cases}$$

  $h(n) \notin A$, since some x are undefined.

3 Question

Consider the set

$$B = \{\mathsf{pair}(n, m) \mid \phi_n(0) = \phi_m(0)\}$$

Prove that $\bar{\mathsf{K}} \leq_m B$.

3.1 Answer

Let's try $\mathsf{K} \leqslant_m \bar{B}$, where

$$\bar{B} = \{\mathsf{pair}(n, m) \mid \phi_n(0) \neq \phi_m(0)\}$$

We define

$$C = \{f(n) | \forall n. f(n) = \#(\lambda x.\phi_n(n)) \wedge \mathsf{dom}(f) = \mathbb{N}\}$$
$$D = \{g(n) | \forall n. g(n) = \#(\lambda x.\phi_n(n) + 1) \wedge \mathsf{dom}(g) = \mathbb{N}\}$$

(RZ: the notation above seems to be more complex than needed. It looks equivalent to

$$C = \{\#(\lambda x.\phi_n(n)) \mid n \in \mathbb{N}\}$$
$$D = \{\#(\lambda x.\phi_n(n) + 1) \mid n \in \mathbb{N}\}$$

Isn't it simpler to avoid defining C, D and directly define

$$f(n) = \#(\lambda x.\phi_n(n))$$
$$g(n) = \#(\lambda x.\phi_n(n) + 1)$$

?)

Suppose we can build $f : \mathsf{K} \to C$, $g : \mathsf{K} \to D$, where $f, g \in \mathcal{R}$, then $h(n) = \mathsf{pair}(f(n), g(n)) \in \mathcal{R}$, pair is an arithmetic operation ($\in \mathcal{R}$), and h is a reduction from K to $\bar{B}$. Then according to the negation lemma, $\bar{\mathsf{K}} \leqslant_m B$.

We show f, g are the reductions from K to C, D respectively.

- For f, take the definition in C, $n \in \mathsf{K} \Rightarrow \forall x.\phi_{f(n)}(x) = \phi_n(n) \Rightarrow f(n) \in C$, o.w. $n \notin \mathsf{K} \Rightarrow \forall x.\phi_{f(n)}(x)$ is undefined, but $\mathsf{dom}(f) = \mathbb{N}$, thus $f(n) \notin C$
- For g, take the definition in D, $n \in \mathsf{K} \Rightarrow \forall x.\phi_{g(n)}(x) = \phi_n(n) + 1 \Rightarrow g(n) \in D$, o.w. due to the same reason, $g(n) \notin D$.

They are indeed the needed reductions. Also

- $n \in \mathsf{K}.\phi_{f(n)}(0) = \phi_n(n) \neq \phi_n(n) + 1 = \phi_{g(n)}(0)$, and $\mathsf{pair}(f(n), g(n)) \in \bar{B}$
- $n \notin \mathsf{K} \Rightarrow \phi_{f(n)}, \phi_{g(n)}$ undefined $\Rightarrow h(n) \notin \bar{B}$

So h is indeed the reduction. Hence proven.