Computability Assignment
Year 2012/13 - Number 10

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1 Question

Prove that the set

$$A = \{n \mid \phi_n(5+n) = 7\}$$

is $\mathcal{RE}$.

1.1 Answer

...

(Answered in previous uploaded file)

2 Question

Prove that the set A defined above is **not** recursive, following the sketch below:

1. Prove that $g(n, x) = 7 \cdot \tilde{\chi}_{\mathsf{K}}(n)$ is a recursive partial function.
2. Prove that $f(n) = \#\left(\lambda x. \begin{cases} 7 & \text{if } n \in \mathsf{K} \\ undefined & \text{otherwise} \end{cases}\right)$ is a recursive total function.
3. Prove that $\chi_{\mathsf{K}}(n) = \chi_A(f(n))$ for all n. (If $n \in \mathsf{K}$ then ... If $n \notin \mathsf{K}$ then ...)
4. Prove that is A were recursive, then the set K would be recursive as well.
5. Conclude that A can not be recursive.

2.1 Answer

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1. The set $K \in \mathcal{RE} \backslash \mathcal{R}$, hence its characteristics funtion $\tilde{\chi}_{\mathsf{K}}$ is a recursive partial function. The composition of this function with the product (recursive total function) returns a recursive partial function. Hence $g(n, x)$ is a recursive partial function.

2. If $n \in \mathcal{K}$ then $f(n) = \#(\lambda x.7)$, since $(\lambda x.7) \in \mathcal{R}$ there exists an index i such that $\varphi_i = (\lambda x.7) \wedge i = \#(\lambda x.7)$, hence $f(n)$ is defined. If $n \notin \mathcal{K}$ then $f(n) = \#(\lambda x. \uparrow)$, since $(\lambda x. \uparrow) \in \mathcal{R}$ there exists an index j such that $\varphi_j = (\lambda x. \uparrow) \wedge j = \#(\lambda x. \uparrow)$, hence $f(n)$ is defined. (RZ: not a good reasoning. Written like this, it seems that f is implemented as "if $n \in \mathsf{K}$ then return i else return j" which is wrong because K is not recursive. It is important here to use the smn theorem and check that the body $b(n, x) = \begin{cases} 7 & \text{if } n \in \mathsf{K} \\ undefined & \text{otherwise} \end{cases}$ is recursive partial, hence h is recrusive total (and injective)) As a consequence $\forall n. f(n) \downarrow$, i.e. f is a total recursive function.

3.

$n \in \mathcal{K} \implies \varphi_n(n) \downarrow \wedge \mathcal{X}_{\mathcal{K}}(n) = 1$

$(n \in \mathcal{K} \implies f(n) = \#(\lambda x.7)) \wedge ((\lambda x.7) \in \mathcal{R} \implies \exists i. \varphi_i = (\lambda x.7)) \implies \varphi_i(5 + i) = 7 \wedge \mathcal{X}_A(n) = \mathcal{X}_A(i) = 1$

$n \notin \mathcal{K} \implies \varphi_n(n) \uparrow \wedge \mathcal{X}_{\mathcal{K}}(n) \uparrow$

$(n \notin \mathcal{K} \implies f(n) = \#(\lambda x. \uparrow)) \wedge ((\lambda x. \uparrow) \in \mathcal{R} \implies \exists j. \varphi_j = (\lambda x. \uparrow)) \implies \varphi_j(5 + j) \uparrow \wedge \mathcal{X}_A(n) = \mathcal{X}_A(j) \uparrow$

Hence $\chi_{\mathsf{K}}(n) = \chi_A(f(n))$.

(RZ: why don't you write $\varphi_{f(n)}(x) = \ldots$? It's not completely clear that i and j stand for $f(n)$ in these two cases)

4. Let's suppose by contraddiction that A is recursive, and consider the following function:

$\psi(x, y) = \begin{cases} 7 & x \in K \\ \uparrow & o.w. \end{cases}$, since $K \in \mathcal{RE}$ the guard check is a partial recursive function (RZ: it's not a function, it's a property/predicate. and it is not even recursive! It is only RE), 1 is recursive (RZ: why 1?) and $\uparrow$is recursive (RZ: this is not enough. When the guard is only RE, to apply the "if-then-else" lemma you need this to be exactly "undef" – just being recursive is not enough (unless the guard is recursive)) ψ is recursive as well.

Then, by S-M-N theorem there exists a recursive (total) function g s.t. $\psi(x, y) = \varphi_{g(x)}(y)$. Then we have:

(RZ: your g was meant to be the f above (not an error, but just a remark))

$x \in K \implies \psi(x, y) = \varphi_{g(x)}(y) = 7 \implies \forall y \in \mathbb{N}. \varphi_{g(x)}(y) = 7 \implies \varphi_{g(x)}(5 + g(x)) = 7 \implies g(x) \in A$

$x \notin K \implies \psi(x, y) = \varphi_{g(x)}(y) \uparrow \implies \forall y \in \mathbb{N}. \varphi_{g(x)}(y) \uparrow \implies \varphi_{g(x)}(5 + g(x)) \uparrow \implies g(x) \notin A$

So we obtained that $K \preceq A$, but $A \in \mathcal{R}$ for hypothesis therefore $K \in \mathcal{R}$.

5. However we know that $K \notin \mathcal{R}$, therefore $A \notin \mathcal{R}$ as well.

3 Question

Prove whether the set $\bar{A}$ is $\mathcal{RE}$, with A as defined above.

3.1 Answer

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(Answered in previously uploaded file)

V'Ger