Computability Assignment
Year 2012/13 - Number 10

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1 Question

Prove that the set

$$A = \{n \mid \phi_n(5+n) = 7\}$$

is $\mathcal{RE}$.

1.1 Answer

Let's write $\tilde{\chi}_A$.

$$\tilde{\chi}_A(n) = \begin{cases} 1 & \text{if } \phi_n(5+n) = 7 \\ \text{undefined} & \text{o.w.} \end{cases}$$

Each part of $\tilde{\chi}_A$ is recursive (RZ: no, the guard is $\mathcal{RE}$, not recursive), thus $\tilde{\chi}_A \in \mathcal{R}$. We show it does work. If $n \in A$, then $\phi_n(5+n) = 7$, it follows $\tilde{\chi}_A(n) = 1$, o.w. $\phi_n(5+n)$ is either not 7 or undefined, we let it be undefined in this case.

2 Question

Prove that the set A defined above is **not** recursive, following the sketch below:

1. Prove that $g(n, x) = 7 \cdot \tilde{\chi}_{\mathsf{K}}(n)$ is a recursive partial function.
2. Prove that $f(n) = \#\left(\lambda x. \begin{cases} 7 & \text{if } n \in \mathsf{K} \\ \text{undefined} & \text{otherwise} \end{cases}\right)$ is a recursive total function.

3. Prove that $\chi_{\mathsf{K}}(n) = \chi_A(f(n))$ for all n. (If $n \in \mathsf{K}$ then ... If $n \notin \mathsf{K}$ then ...)
4. Prove that if A were recursive, then the set K would be recursive as well.
5. Conclude that A can not be recursive.

2.1 Answer

1. We expand $g(n, x)$.

$$g(n,x) = \begin{cases} 7 & \text{if } n \in \mathsf{K} \\ \text{undefined} & \text{o.w.} \end{cases}$$

   $\mathsf{K} \in \mathcal{RE} \Rightarrow \tilde{\chi}_{\mathsf{K}} \in \mathcal{R}$. Each part of $g(n, x)$ is recursive (RZ: no, the guard is $\mathcal{RE}$, not recursive), thus $g(n, x) \in \mathcal{R}$.

2. Take any i s.t. $\varphi_i = g$, we observe that $f(n) = s_1^1(i, n)$, due to s-m-n theorem, such an f is total and recursive.
3. 1) If $n \in \mathsf{K}$, i.e. $\tilde{\chi}_{\mathsf{K}}(n) = 1$, then $\tilde{\chi}_A(f(n)) = \tilde{\chi}_A(\#(\lambda x.7))$, and $\#(\lambda x.7)$ is the (RZ: an) index of the constant program (RZ: function) 7 on any input, we see that $\#(\lambda x.7) \in A$. Thus $\tilde{\chi}_A(f(n)) = 1$.
   2) If $n \notin \mathsf{K}$, then $\tilde{\chi}_A(f(n)) = \chi_A(\#(\lambda x.\Omega))$, we have the index of a program never halts, apparently $\#(\lambda x.\Omega) \notin A$, thus $\tilde{\chi}_A(f(n)) = \text{undefined} = \tilde{\chi}_{\mathsf{K}}(n)$.
4. Suppose A is recursive, then $\chi_A(n) = 0$ if $n \notin A$ (RZ: irrelevant?), we can build $\chi_{\mathsf{K}}(n) = \chi_A(f(n)) \in \mathcal{R}$, concluding $\mathsf{K} \in \mathcal{R}$. This is not true.
5. By contradiction, use 4.

3 Question

Prove whether the set $\bar{A}$ is $\mathcal{RE}$, with A as defined above.

3.1 Answer

It follows $A \notin \mathcal{R}$, that is $\bar{A} \notin \mathcal{RE}$, otherwise, suppose $\bar{A} \in \mathcal{RE}$, we already have $A \in \mathcal{RE}$, then $A \in \mathcal{R}$ (by parallelly running the semi-characteristic functions of A and $\bar{A}$, we can get a characteristic function for A). This is a contradiction.