Computability Assignment
Year 2012/13 - Number 10

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1 Question

Prove that the set

$$A = \{n \mid \phi_n(5+n) = 7\}$$

is $\mathcal{RE}$.

1.1 Answer

...

Set A has a semi-verifier (RZ: semi-characteristic function)

$S_A(x) = \begin{cases} 1 & if\ \varphi_x(5+x) = 7 \\ \uparrow & else \end{cases}$

which is obviously recursive. (RZ: this actually relies on A being $\mathcal{RE}$)

Hence $A \in \mathcal{RE}$, and this means that $A \preceq K$. (TRIVIA:K is complete, i.e. $\forall S \in \mathcal{RE}. S \preceq K$)

Let's verify if $K \preceq A$. Define a

$\psi(x,y) = \begin{cases} 7 & \varphi_x(x)\downarrow\ (hence\ x \in K) \\ \uparrow & o.w. \end{cases}$

which is once again obviusly recursive, then by S-M-N theorem there exists a recursive (total) function g s.t. $\psi(x,y) = \varphi_{g(x)}(y)$.

But then we have that:

$x \in K \iff \varphi_x(x)\downarrow \Longrightarrow \forall y. \varphi_{g(x)}(y) = 7 \Longrightarrow \varphi_{g(x)}(5+g(x)) = 7 \Longrightarrow g(x) \in A$

$x \notin K \iff \varphi_x(x)\uparrow \Longrightarrow \forall y. \varphi_{g(x)}(y)\uparrow \Longrightarrow \varphi_{g(x)}(5+g(x))\uparrow \Longrightarrow g(x) \notin A$

Since we have found that $K \preceq A$, by $A \in \mathcal{RE} \wedge A \preceq K \wedge K \preceq A$ we conclude that A is creative and complete. (info: MYHILL THM, various lemmas)

(ANSWER 3.1) Since we have that $K \preceq A \iff \overline{K} \preceq \overline{A}$ and the property $B\,productive \wedge B \preceq A \implies A\,productive$, we get that $\overline{A} = \{n \mid \varphi_n(5+n) \neq 7\}$ is productive. (i.e. $\overline{A} \notin \mathcal{RE}$).

NOTE:

"Obviously recursive" means - for example in the first case - "Take program with Goedel index x and run it over input $5+x$. If it halts and it outputs 7, return 1. If it halts but it doesn't output 7, loop forever. Otherwise just wait forever for program with index x to terminate". One could easily provide pseudo-code for this program, but IMHO is unnecessary.

(RZ: ok, pseudo-code is not needed, but you have to provide some description as in the lines above – just writing "obvious" will not convince me at an exam)

2 Question

Prove that the set A defined above is **not** recursive, following the sketch below:

1. Prove that $g(n,x) = 7 \cdot \tilde{\chi}_{\mathsf{K}}(n)$ is a recursive partial function.
2. Prove that $f(n) = \#\left(\lambda x. \begin{cases} 7 & \text{if } n \in \mathsf{K} \\ undefined & \text{otherwise} \end{cases}\right)$ is a recursive total function.
3. Prove that $\chi_{\mathsf{K}}(n) = \chi_A(f(n))$ for all n. (If $n \in \mathsf{K}$ then ... If $n \notin \mathsf{K}$ then ...)
4. Prove that is A were recursive, then the set K would be recursive as well.
5. Conclude that A can not be recursive.

2.1 Answer

...

(Will be uploaded by the end of the week in a new file)

3 Question

Prove whether the set $\bar{A}$ is $\mathcal{RE}$, with A as defined above.

3.1 Answer

...

See answer 1.1

V'GER