Computability Assignment
Year 2012/13 - Number 9

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1 Question

Assume that f is a recursive partial function satisfying the following property:

$$\forall n, m \in \mathbb{N}.\ f(2^n 3^m) = \begin{cases} m+1 & \text{if } \phi_n(m) = 0 \\ 0 & \text{otherwise} \end{cases}$$

(Note that the above makes no guarantees on e.g. what $f(7)$ actually is).

Prove that:

1. The set $A = \{n \mid \phi_n(5 \cdot n) = 0\}$ is recursive.
2. (Harder, feel free to skip it)
   The set $B = \{n \mid \phi_n(2) = 5\}$ is recursive.

(Note: actually f is a non recursive function, and A, B are non recursive sets. Still, I'm interested in how one proves the above portion of a reduction argument.)

1.1 Answer

1. Since for all $n, m \in \mathbb{N}$ the property is satisfied, we pick $m = 5n$. Thus we can write

   $$f(2^n 3^{5n}) = \begin{cases} 5n+1 & \text{if } \phi_n(5n) = 0 \\ 0 & \text{otherwise} \end{cases}$$

   We define a set $M = \{5n+1 \,|\, \forall n \in \mathbb{N}\}$, it follows that $M \in \mathcal{R}$ ($\chi_M(n) = \text{if}\,(n \,\mathbf{mod}\, 5 = 1)\,\text{then}\,1\,\text{else}\,0$). We use χ_M to build the result for A. That is, $\chi_A(n) = \chi_M(f(2^n 3^{5n})), \forall n \in \mathbb{N}$, i.e. assuming the hyphothesis, $A \in \mathcal{R}$.

We show that χ_A does work. 1) If $n \in A$, we have $\phi_n(5n) = 0$, thus $f(2^n 3^{5n}) = 5n + 1$, $\chi_M(5n + 1) = 1$. 2) If $n \notin A$, we have $\phi_n(5n) \neq 0$, thus $f(2^n 3^{5n}) = 0$, $\chi_M(0) = 0$.
(RZ: OK, $M = \mathbb{N} \setminus \{0\}$ could have been easier.)

2. Following the above spirit, let's write

$$f(2^n 3^2) = \begin{cases} 3 & \text{if } \phi_n(2) = 0 \\ 0 & \text{otherwise} \end{cases}$$

We first build the result for $C = \{n \,|\, \phi_n(2) = 0\}$. i.e., $\chi_3(f(2^n 3^2)) = \chi_C(n)$, where χ_3 is the characteristic function for the constant 3. One can verify χ_C does work following the above proof on χ_A. Now we utilize χ_C to obtain χ_B. Then, we build a bijection (RZ: ??) $t : B \leftrightarrow C$, 1) $g(n)(x) = \phi_n(x) + 5, \forall x \in \mathbb{N}, \forall n \in C$, we write $g(n)(x) = \phi_{n'}(x)$ (renaming) (RZ: I can't see what you are doing). Thus, $B = \{n' \,|\phi_{n'}(x) = g(n)(x), n \in C\}$. In the same style, we write 2) $h(n')(x) = \phi_{n'}(x) - 5, \forall x \in \mathbb{N}, \forall n' \in B$, we write $h(n')(x) = \phi_n(x)$, thus $C = \{n \,|\phi_n(x) = h(n')(x), n' \in B\}$. Finally, $\chi_B(n) = \chi_C(t(n))$.
We show that χ_B works. 1) If $n \in B$, then $t(n) \in C$, $\chi_C(t(n)) = 1$. 2) If $n \notin B$, then $t(n) \notin C$, thus $\chi_C(t(n)) = 0$.