Computability Assignment
Year 2012/13 - Number 9

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1 Question

Assume that f is a recursive partial function satisfying the following property:

$$\forall n, m \in \mathbb{N}.\ f(2^n 3^m) = \begin{cases} m+1 & \text{if } \phi_n(m) = 0 \\ 0 & \text{otherwise} \end{cases}$$

(Note that the above makes no guarantees on e.g. what $f(7)$ actually is).

Prove that:

1. The set $A = \{n \mid \phi_n(5 \cdot n) = 0\}$ is recursive.
2. (Harder, feel free to skip it)
   The set $B = \{n \mid \phi_n(2) = 5\}$ is recursive.

(Note: actually f is a non recursive function, and A, B are non recursive sets. Still, I'm interested in how one proves the above portion of a reduction argument.)

1.1 Answer

...

1.

We shall define a verifier v_A for A:

$v_A(x) = \begin{cases} 1 & if\ f(2^x 3^{5x}) = 5x+1 \\ 0 & o.w. \end{cases}$

(RZ: don't confuse a characteristic function with a verifier (which is a program implementing the char.function))

The verifier of A is recursive under the assumption of f being recursive (which is not).

(RZ: ok, note that $f(2^n 3^{5n}) = 5x + 1$ is recursive because f is recursive and $f(2^n 3^m)$ is defined. If the latter could be undefined, then that predicate would not be recursive.)

Let's check that it's also properly defined:

$v_A(x) = 1 \iff f(2^x 3^{5x}) = 5x + 1 \iff \varphi_x(5x) = 0 \iff x \in A$

$v_A(x) = 0 \iff f(2^x 3^{5x}) = 0 \iff \varphi_x(5x) \neq 0 \iff x \notin A$

(RZ: OK. Formally, the first chain of $\iff$ is enough. In an exam, I'd expect more justification – here it's OK.)

2.

Define

$$\psi(x, y) = \begin{cases} 0 & if\ \varphi_x(y) = 5 \\ \uparrow & o.w. \end{cases}$$

it's easy to see that ψ is recursive (RZ: ok, in an exam, state why (e.g. using the "if-then-else" lemma on RE predicates)), hence by S-M-N theorem there exists a (total) recursive function g s.t. $\psi(x, y) = \varphi_{g(x)}(y)$.

Now we can define v_B for B in the following way:

$$v_B(x) = \begin{cases} 1 & if\ f(2^{g(x)} 3^2) = 3 \\ 0 & o.w. \end{cases}$$

Again, the verifier of B is recursive under the assumption of f being recursive (which is not). Let's check that it's also properly defined:

$v_B(x) = 1 \iff f(2^{g(x)} 3^2) = 3 \iff \varphi_{g(x)}(2) = 0 \iff \varphi_x(2) = 5 \iff x \in B$

$v_B(x) = 0 \iff f(2^{g(x)} 3^2) = 0 \iff \varphi_{g(x)}(2) \uparrow \iff \varphi_x(2) \neq 5 \iff x \notin B$

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