Computability Assignment
Year 2012/13 - Number 8

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1 Question

Prove that the following set is **not** λ-definable.

$$A = \{\#M \mid \exists n \in \mathbb{N}.\ M\ulcorner n \urcorner =_{\beta\eta} \ulcorner 5 \urcorner\}$$

1.1 Answer

- (sem. closed) $\#M \in A \Rightarrow \exists n.M \ulcorner n \urcorner =_{\beta\eta} \ulcorner 5 \urcorner, \forall N.M =_{\beta\eta} N \Rightarrow M \ulcorner n \urcorner =_{\beta\eta} N \ulcorner n \urcorner =_{\beta\eta} \ulcorner 5 \urcorner \Rightarrow \#N \in A$
- (not empty) $\#(\lambda n.\ulcorner 5 \urcorner) \in A$
- (not $\mathbb{N}$) $\#(\lambda n.\ulcorner 0 \urcorner) \notin A$

By Rice's theorem, A is not λ-definable.

2 Question

Prove that the following set is semantically closed. Then, prove that it **is** λ-definable.

$$A = \{\#M \mid \forall N \in \Lambda.\ N\,M =_{\beta\eta} \mathbf{I}\}$$

2.1 Answer

- (sem. closed) $\#M \in A \Rightarrow \forall N \in \Lambda. N\, M =_{\beta\eta} \mathbf{I}, \forall M' \in \Lambda. M =_{\beta\eta} M' \Rightarrow NM =_{\beta\eta} NM' =_{\beta\eta} \mathbf{I}$
  (RZ: you don't need this!)

- We prove $A = \varnothing$. Suppose $\exists M. \forall N \in \Lambda. NM =_{\beta\eta} \mathbf{I}$, Let $N' = \lambda m. \mathbf{K}\, \Omega\, m \in \Lambda$, we have $N'M =_{\beta\eta} \Omega$. Hence $V_A = \lambda n. \mathbf{F}$.

Note.

The following exercise is harder. Feel free to skip it.

3 Question

Prove **whether** the following set is λ-definable.

$$A = \{\#M \mid M \ulcorner M \urcorner =_{\beta\eta} M\}$$

(Note: there is at least one simple solution to this. You do not need to try huge formulae for this.)

3.1 Answer

Due to the result we had during the lecture, this set is not semantically closed (from $M =_{\beta\eta} N$, we can hardly have $\ulcorner M \urcorner =_{\beta\eta} \ulcorner N \urcorner$). It seems a hard set.

Let's suppose it is λ-definable. Then we can write a verifier V_A.

Thus $B = \{\#M \,|\, \mathbf{K}M =_{\beta\eta} M\}$ is λ-defined by

$$V_B = \lambda n. V_A(\mathbf{App}\,\mathbf{K}\,n)$$

(RZ: should be $V_B = \lambda n. V_A(\mathbf{App}\,\mathbf{K}\,(\mathbf{Num}\,n))$)

We show that such a V_B does work. Indeed,

$$\mathbf{K}\, M \ulcorner \mathbf{K}M \urcorner =_{\beta\eta} \mathbf{K}M \Rightarrow \mathbf{K}M =_{\beta\eta} M$$

(RZ: ok, a bit more details about why V_B works (in the exam, at least))

However, B is not λ-definable according to Rice's theorem:

- B is sem. closed. $M =_{\beta\eta} N \wedge \mathbf{K}M =_{\beta\eta} M \Rightarrow \mathbf{K}M =_{\beta\eta} \mathbf{K}N =_{\beta\eta} M =_{\beta\eta} N$

- B is not empty, by Tarski, take $M = \Theta\mathbf{K}$.

- B is not $\mathbb{N}$, $\mathbf{KK} \neq_{\beta\eta} \mathbf{K}$

We reach a contradiction, hence A is not λ-definable.