Computability Assignment
Year 2012/13 - Number 8

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1 Question

Prove that the following set is **not** λ-definable.

$$A = \{\#M \mid \exists n \in \mathbb{N}.\ M\ulcorner n\urcorner =_{\beta\eta} \ulcorner 5\urcorner\}$$

1.1 Answer

...

To proove that A is not λ-definable, one can use the Theorem of Rice, which requires the following three conditions to hold:

1) A is semantically closed

$\#M \in A \wedge M =_{\beta\eta} N$

$\Longrightarrow \exists n \in \mathbb{N}.M\ulcorner n\urcorner =_{\beta\eta} \ulcorner 5\urcorner \wedge M =_{\beta\eta} N$

$\Longrightarrow \exists n \in \mathbb{N}.N\ulcorner n\urcorner =_{\beta\eta} \ulcorner 5\urcorner$

$\Longrightarrow \#N \in A$, Ok

2) $A \neq \emptyset$

$\#(\lambda x_0.\ulcorner 5\urcorner) \in A$

3) $A \neq \mathbb{N}$

$\#\Omega \notin A$

Hence, as a matter of fact, the set A is not recursive // λ-definable.

2 Question

Prove that the following set is semantically closed. Then, prove that it **is** λ-definable.

$$A = \{\#M \mid \forall N \in \Lambda.\ N\, M =_{\beta\eta} \mathbf{I}\}$$

2.1 Answer

...

1) Semantically Closed (RZ: you don't need this)
$\#M \in A \wedge M =_{\beta\eta} P$
$\Longrightarrow \forall N \in \Lambda . NM =_{\beta\eta} I \wedge M =_{\beta\eta} P$
$\Longrightarrow \forall N \in \Lambda . NP =_{\beta\eta} I$
$\Longrightarrow \#P \in A$
2) Prove that it is λ-definable
Note that it does not exist any $\#M$ such that $(K\Omega)M =_{\beta\eta} I$. Therefore $A = \emptyset$ and it is obviously recursive / λ-definable.

Note.

The following exercise is harder. Feel free to skip it.

3 Question

Prove **whether** the following set is λ-definable.

$$A = \{\#M \mid M \ulcorner M \urcorner =_{\beta\eta} M\}$$

(Note: there is at least one simple solution to this. You do not need to try huge formulae for this.)

3.1 Answer

...

Let's define
$V_B \ulcorner M \urcorner = (\lambda x. V_A \ulcorner (App \ulcorner K \urcorner\, x) \urcorner) \ulcorner M \urcorner = V_A \ulcorner (App \ulcorner K \urcorner\, M) \urcorner = V_A \ulcorner KM \urcorner$
(RZ: formally, you can not use x under the corners $\ulcorner x \urcorner$, you need to exploit Lam, App, Num instead.)
so that we can get rid of the nasty source code reference. The latter step behaves in the following way:
$\ulcorner KM \urcorner \in A \Longleftrightarrow KM \ulcorner KM \urcorner =_{\beta\eta} KM \Longleftrightarrow M =_{\beta\eta} KM$
Let's see what V_B gives us: $B = \{\#M |\ M =_{\beta\eta} KM\}$. We are lucky and the set looks not recursive, which is exactly what is needed to use the reduction argument approach and state that A is not recursive as well. So let's prove that this set is not recursive using Rice Theorem:
- Semantically Closed:
$\#M \in B \wedge M =_{\beta\eta} N$

$\Longrightarrow M =_{\beta\eta} KM \wedge M =_{\beta\eta} N$
$\Longrightarrow N =_{\beta\eta} KN$
$\Longrightarrow \#N \in B$

- $B \neq \emptyset$: (here I may be wrong in the usage of $E/\ Num$..?)

Let's take $M = \Theta F$, with $F = \lambda f.\, E\,(Lam \ulcorner\ulcorner 0 \urcorner\urcorner\,(Num\, f))$, then:

(RZ: $Num\ f$ doesn't work exactly as this. $M = \Theta K = K(\Theta K) = KM$ is much simpler (and correct))

rh: $M = \Theta F = E\,(Lam \ulcorner\ulcorner 0 \urcorner\urcorner \ulcorner \Theta F \urcorner) = E \ulcorner \lambda x_0.\Theta F \urcorner = \lambda x_0.\Theta F$

lh: $KM = (\lambda xy.x)\,(\Theta F) = \lambda y.\Theta F$

The left/right handsides are $\beta\eta - equivalent$ up to α-conversions, therefore $\#(\Theta F) \in B$.

- $B \neq \mathbb{N}$:

Take $M = I$, then $KI = \lambda y.I \neq I$, hence $\#I \notin B$.

V'Ger