Computability Assignment
Year 2012/13 - Number 8

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1 Question

Prove that the following set is **not** λ-definable.

$$A = \{\#M \mid \exists n \in \mathbb{N}.\ M\ulcorner n\urcorner =_{\beta\eta} \ulcorner 5\urcorner\}$$

1.1 Answer

To prove that we can use the Rice theorem. The numbers 1, 2, 3 correspond to the requirements of the theorem (as we saw in class):

1. First we need to prove that the set A is semantically closed. Take $M \in A$ and $M =_{\beta\eta} N$ for all programs M, N. So $M\ulcorner n\urcorner =_{\beta\eta} \ulcorner 5\urcorner$ but this implies that also $N\ulcorner n\urcorner =_{\beta\eta} \ulcorner 5\urcorner \forall n$ and so $\#N \in A$.

2. Second we need to prove that A is not empty. Just take $\#I$: it belongs to the set because $I\ulcorner n\urcorner =_{\beta\eta} \ulcorner 5\urcorner$ for some n (in particular with $n = 5$)

3. Third we need to prove that $A \neq \mathbb{N}$. Take, for example, all the programs that just discard the input $\ulcorner n\urcorner$ and return always the same value like $\lambda x.\ulcorner 1\urcorner$. They are not part of A so the property is verified.

With these properties we can state, by the Rice theorem, that the set is not λ-definable.

2 Question

Prove that the following set is semantically closed. Then, prove that it **is** λ-definable.

$$A = \{\#M \mid \forall N \in \Lambda.\ N\, M =_{\beta\eta} \mathbf{I}\}$$

2.1 Answer

First, we need to prove that the set is semantically closed. Take $M \in A$ and $M =_{\beta\eta} O$ for programs M, O. So $\forall N \in \Lambda.NM =_{\beta\eta} I$ but we can substitute M with O and by that we have $\forall N \in \Lambda.NO =_{\beta\eta} I$ and so $\#O \in A$. Now, consider the 3 properties of the Rice theorem. If all of them are true, the set is not λ-definable. We know that the first one is verified (set is semantically closed) and so either $A \neq \emptyset$ or $A \neq \mathbb{N}$ must be false (clearly they cannot be both false at the same time). Then take, for example, $N, M = Succ$: (RZ: this example is a bit complex) in this case the program is not equal to the identity and so $\#M \notin A$; this implies that $A \neq \mathbb{N}$ and so it must be that $A = \emptyset$ (otherwise for Rice theorem A will be not λ-definable). This alone cannot ensure that the set is λ-definable (because false assumptions do not implies false implications) but if the set is empty we can construct a verifier for it that always return false and so the set is λ-definable.

(RZ: do not involve Rice in your proof. Just proving $A = \emptyset$ is enough to claim that A is λ-definable. Also, you need to **prove** that A is empty!)

Note.

The following exercise is harder. Feel free to skip it.

3 Question

Prove **whether** the following set is λ-definable.

$$A = \{\#M \mid M\ulcorner M \urcorner =_{\beta\eta} M\}$$

(Note: there is at least one simple solution to this. You do not need to try huge formulae for this.)

3.1 Answer

Write your answer here.