Computability Assignment
Year 2012/13 - Number 7

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1 Question

Prove that the following set is not λ-definable.

$$A = \{\#M \mid M \text{ has a } \beta\text{-normal form}\}$$

(Hint: show that, if A were λ-definable, then also K_λ would be λ-definable, hence obtaining a contradiction.)

1.1 Answer

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TRIVIA: $K_\lambda = \{\#M \mid M\ulcorner M\urcorner\, has\, a\, \beta - normal\, form\}$ (Definition 140, p. 51 Notes)

Let's suppose that there exists a verifier V_A for the set A s.t.

$V_A(\#M) = \begin{cases} 1 & M\, has\, a\, \beta - normal\, form \\ 0 & o.w. \end{cases}$

Then we could construct a verifier V_{K_λ} in the following way:

$V_{K_\lambda}(\#M) = \begin{cases} 1 & M\ulcorner M\urcorner\, belongs\, to\, A \\ 0 & o.w. \end{cases}$

Intuitively, we have used the property that with the expression $M\ulcorner M\urcorner$ we are fixing a chosen input to a chosen program, which in turns continues to be a program (with less input variables). Since the verifier V_A can tell us whether a program has a $\beta - normal\, form$, we can use it to build the verifier for the set K_λ. This is a contraddiction, since the set set K_λ can have only a partial verifier, which is mandatory since it's a recursively enumerable set strictly not recursive.

2 Question

Let A be a λ-definable set. Prove that

$$B = (A \cup \{b_1, \dots, b_n\}) \setminus \{c_1, \cdots, c_m\}$$

is also λ-definable.

(Hint: do not reinvent the results we saw in class, just apply them.)

2.1 Answer

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TRIVIA: by theorem, a $\lambda - definable$ set is closed under union, complement and intersection operation with a single element. (Lemma 133, p. 49 Notes)

Given fixed $n, m \in \mathbb{N}$, we can reuse the theorem property and prove that B is $\lambda - definable$ using the following construction:

$B = ((((((((((A \cup \{b_1\}) \cup \{b_2\}) ...) \cup \{b_n\}))\backslash\{c_1\})\backslash\{c_2\}) ...)\backslash\{c_m\}))$

Then, by applying a union / complementation with a single element in a left-to-right order we are able to reuse the theorem and prove that each and every intermediate set (plust the last one) is $\lambda - definable$ as we wished.

3 Question

Let A be a **non** λ-definable set. Prove that

$$B = (A \cup \{b_1, \dots, b_n\}) \setminus \{c_1, \cdots, c_m\}$$

is also **non** λ-definable.

(Hint: prove the contrapositive. That is, prove that if B were λ-definable, then also A would be such.)

3.1 Answer

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The Hint says really everything. Let's write:

$A = ((((((((((B \cup \{c_m\}) \cup \{c_{m-1}\}) ...) \cup \{c_1\}))\backslash\{b_n\})\backslash\{b_{n-1}\}) ...)\backslash\{b_1\}))$

by the same theorem mentioned above, the assumption of B being $\lambda - definable$ follows in the necessary consequence that also A is $\lambda - definable$. Which contraddicts the property we already know. Hence B must be a **non** $\lambda - definable$ set.

(RZ: correct idea. Note that the above equation is not completely true in the general case. For instance if $A = \mathbb{N}$, $B = (A \cup \{3, 4, 5\}) \setminus \{4, 5, 6\} = \mathbb{N} \setminus \{4, 5, 6\}$ then we do not have $A = (B \cup \{4, 5, 6\}) \setminus \{3, 4, 5\} = \mathbb{N} \setminus \{3, 4, 5\}$. Before performing what you did you need to reduce $\{b_i, \dots\}$ and $\{c_i, \dots\}$ to a minimal set, by removing e.g. those b_i which occur in A or in $\{c_i\}$, and removing those c_i not belonging to $A \cup \{b_i, \dots\}$)

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