Computability Assignment
Year 2012/13 - Number 5

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1 Question

(I am re-proposing this exercise since only a few students solved it. This exercise is rather important, since it involves a reasoning which frequently appeared in past exam questions. While we shall see more examples of these concepts in class, it would be useful to start exercising on that. If you have already sumbitted an answer, skip this and do *not* resubmit your answer please.)

Let $\mathcal{F}$ be the set of partial functions $\{f \in (\mathbb{N} \rightsquigarrow \mathbb{N}) | \forall x \in \mathbb{N}.\ f(2 \cdot x) = x\}$.

- Define two distinct partial functions f_1, f_2 which belong to $\mathcal{F}$. (I.e, provide two such examples.)
- Define two distinct partial functions g_1, g_2 which do *not* belong to $\mathcal{F}$. (I.e, provide two such examples.)
- Define a partial function $f \in \mathcal{F}$, and consider the set of its *finite* restrictions $\mathcal{G} = \{g \in (\mathbb{N} \rightsquigarrow \mathbb{N}) | g \subseteq f\ \wedge\ \mathsf{dom}(g) \text{ finite}\}$.
  - Define two distinct partial functions h_1, h_2 which belong to $\mathcal{G}$. (I.e, provide two such examples.)
  - Prove whether $\mathcal{F} \cap \mathcal{G} = \emptyset$.

1.1 Answer

Done previously.

2 Question

Consider the following function:

$$f(n) = \sum_{i=0}^{n} i^2 + i$$

Write a FOR loop implementing f, then translate it in the λ-calculus as program F_1.

Then, write a recursive Java-like function implementing f, and translate it in the λ-calculus as program F_2.

2.1 Answer

1. ```
   s := 0;
   for i := 0 to n do {
           s := s + i*i + i;
   }
   ```

   In λ-calculus, we use pairs to save the result of sums.

   $$\begin{aligned} G &= \lambda p.\mathbf{Cons}\ (\mathbf{Add}\ (\mathbf{Fst}\ p)\ (\mathbf{Add}\ (\mathbf{Snd}\ p)\ (\mathbf{Mul}\ (\mathbf{Snd}\ p)\ (\mathbf{Snd}\ p))))\ (\mathbf{Succ}\ (\mathbf{Snd}\ p)) \\ F &= \lambda n.n\ G\ (\mathbf{Cons}\ \ulcorner 0 \urcorner\ \ulcorner 0 \urcorner) \end{aligned}$$

   (RZ:$F = \lambda n.Fst\ (Succ\ n\ G\ (\mathbf{Cons}\ \ulcorner 0 \urcorner\ \ulcorner 0 \urcorner))$

2. ```
   int f(int n){
           if(n >= 0)
                   return n*n + n + f(n-1);
           return 0;
   }
   ```

   We construct it in λ-calculus

   $$\begin{aligned} G &= \lambda gn.\mathbf{Leq}\ n\ \ulcorner 0 \urcorner \ulcorner 0 \urcorner (\mathbf{Add}\ (\mathbf{Add}\ n\ (\mathbf{Mul}\ n\ n))\ (g\ g\ (\mathbf{Pred}\ n))) \\ F &= G\ G \end{aligned}$$

   (RZ: watch out, $n \leq 0$ means $n = 0$ on naturals.)

3 Question

Consider the following function:

$$f(n) = \begin{cases} x^2 + y & \text{if } n = \mathsf{pair}(\mathsf{inL}(x), y) \\ x + 4 \cdot y & \text{if } n = \mathsf{pair}(\mathsf{inR}(x), y) \end{cases}$$

Convince yourself that f is defined for all naturals n, i.e. it is total.

Write a λ-term implementing function f, exploiting the programs $Pair, Proj1, Proj2, InL, InR, Case, \ldots$ we saw in class (also defined in the notes).

3.1 Answer

$$
\begin{aligned}
L &= \lambda mp.\mathbf{Add}\ (\mathbf{Mul}\ p\ p)\ \ulcorner \mathsf{Proj2}(m) \urcorner \\
R &= \lambda mq.\mathbf{Add}\ q\ (\mathbf{Mul}\ \ulcorner \mathsf{Proj2}(m) \urcorner\ \ulcorner 4 \urcorner) \\
F &= \lambda n.\mathbf{Case}\ \ulcorner \mathsf{Proj1}(n) \urcorner\ (L\ n)\ (R\ n)
\end{aligned}
$$

(RZ: don't use functions under corners: e.g. write $(Proj1\ n)$ instead of $\ulcorner \mathsf{Proj1}(n) \urcorner$)