Question 1. Let $\mathcal{F}$ be the set of parital functions $\{f \in N \rightsquigarrow N | \forall x \in \mathbb{N}. f(2x) = x\}$

- Define two distinct partial functions f_1, f_2 which belong to $\mathcal{F}$
  i) Let $f_1 \in \mathcal{F}$ be such that $\forall n \in \mathbb{N}$,

$$f_1(n) = \begin{cases} n/2 & ,\text{if } n \text{ is even} \\ n & ,\text{otherwise} \end{cases}$$

  ii) Let $f_2 \in \mathcal{F}$ be such that $\forall n \in \mathbb{N}$,

$$f_2(n) = \begin{cases} n/2 & ,\text{if } n \text{ is even} \\ \text{undefined} & ,\text{otherwise} \end{cases}$$

- Define two distinct partial functions g_1, g_2 that do not belong to F
  i) g_1 is the empty function $\emptyset$, which is a subset of $\mathbb{N} \times \mathbb{N}$, for which for any $n \in \mathbb{N}$, $g(n)$ is undefined
  ii) Let g_2 be such that $\forall n \in \mathbb{N}$,

$$g_2(n) = \begin{cases} n & ,\text{if } n = 0 \\ \text{undefined} & ,\text{otherwise} \end{cases}$$

- Define a partial function $f \in \mathcal{F}$, and consider the set of its finite restrictions $\mathcal{G} = \{g \in \mathbb{N} \rightsquigarrow \mathbb{N} | g \subseteq f \wedge dom(g)\ finite\}$
  Let $f \in \mathcal{F}$ be such that $\forall n \in \mathbb{N}$,

$$f(n) = \begin{cases} n/2 & ,\text{if } n \text{ is even} \\ \text{undefined} & ,\text{otherwise} \end{cases}$$

  - Define two partial functions h_1, h_2 that belongs to $\mathcal{G}$
    i) Let $h_1 : \{0\} \to \mathbb{N}$ be defined such that $\forall n \in \mathbb{N}$,

$$h_1(n) = \begin{cases} n/2 & ,\text{if } n = 0 \\ \text{undefined} & ,\text{otherwise} \end{cases}$$

    ii) Let $h_2 : \{2\} \to \mathbb{N}$ be defined such that $\forall n \in \mathbb{N}$,

$$h_2(n) = \begin{cases} n/2 & ,\text{if } n = 2 \\ \text{undefined} & ,\text{otherwise} \end{cases}$$

  - Prove whether $\mathcal{F} \cap \mathcal{G} = \emptyset$
    $\mathcal{F} \cap \mathcal{G} = \emptyset$, since by definition, for any $f \in \mathcal{F}, dom(f) \supseteq \{0, 2, ..\}$, i.e. $dom(f)$ atleast contains the set of even numbers which is a infinite set, where as by definition any $g \in \mathcal{G}, dom(g)$ is finite. Since there exists no $g \in \mathcal{G}$ such that $dom(g)$ that contains the infinite set of even numbers, $\mathcal{F} \cap \mathcal{G} = \emptyset$

Definition 1. *Let $\mathcal{R}$ be the set of inference rules over elements of a set A. Then R induces a function $\hat{R} : \mathcal{P}(A) \to \mathcal{P}(A)$ given by*

$$\hat{R}(X) = \{y | \exists \frac{x_1, ..., x_n}{z} \in \mathcal{R} \wedge y = z \wedge \forall i \in \{1, ..., n\}. x_i \in X\}$$

Question 2. Let m, n range over $\mathbb{N}$, consider the following set of inference rules $\mathcal{R}$

$$\frac{n \ m}{n.m} \qquad \frac{}{1} \qquad \frac{n}{n.2}$$

and the sets

$$E = \{2.n | n \in \mathbb{N}\} \qquad O = \{2.n + 1 | n \in \mathbb{N}\}$$

Then answer the following questions:

1. State whether $\hat{R}(O) \subseteq O$
   No. because $1 \in \hat{R}(O)$, where as the least element in O is 2.

2. State whether $O \subseteq \hat{R}(O)$. Since $O = \{2, 4, 6, ...\}$, where as $2 \notin \hat{R}(O)$. The answer is NO

3. State whether $\hat{R}(E) \subseteq E$.
   Since $E = \{0, 2, 4, ...\}$ and since $1 \in \hat{R}(R)$, where as $1 \notin E$, the answer is NO.

4. State whether $E \subseteq \hat{R}(E)$. Since E is the set of even numbers and since the application of inference rule $\frac{n}{2.n}$ on E produces E it self. The answer is YES.

5. State whether $\hat{R}(\mathbb{N}) \subseteq \mathbb{N}$. Since $\mathbb{N}$ is closed w.r.t the set of results in the consequence of each operator namely $n.m, 1, n.2$, $\hat{R} \subseteq \mathbb{N}$. The answer is YES.

6. State whether $\mathbb{N} \subseteq \hat{R}(\mathbb{N})$. Since the application of inference rule $\frac{n \quad m}{n.m}$ on $n = 1$, for all $m \in \mathbb{N}$ produces $\mathbb{N}$ it self. The answer is YES.

7. State whether $\hat{R}(E \cup \{1\}) \subseteq E \cup \{1\}$. Since $E \cup \{1\} = \{0, 1, 2, 4, 6, ...\}$. Since the application rules $\frac{m \quad n}{m.n}$ and $\frac{n}{2.n}$ on the set $E \cup \{1\}$ produces only the set E , where as $\frac{}{1}$ produces only $\{1\}$, hence $\hat{R}(E) \cup \{1\} = E \cup \{1\}$
8. Characterize the minimum fix-point of $\hat{R}$, i.e $\bigcap\{X | \hat{R}(X) = X\}$. Since by previous answers it follows that, the minimum fix-point is $E \cup \{1\}$
9. Characterize the maximum fix-point of $\hat{R}$, i.e $\bigcup\{X | \hat{R}(X) = X\}$. Since we already proved that $\hat{R}(\mathbb{N}) \subseteq \mathbb{N}$. The maximum fix-point is N