

Computability Assignment

Year 2012/13 - Number 4

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Please do not submit a file containing only the answers; edit this file, instead, filling the answer sections.

1 Preliminaries

A partial function g is said to be a *restriction* of a partial function f , written $g \subseteq f$ iff

$$\forall x \in \text{dom}(g). g(x) = f(x)$$

Note: this notation “overloads” the symbol \subseteq . Indeed, we shall write $A \subseteq B$ to express a subset relation between two sets, and $g \subseteq f$ to express a restriction relation between two functions.

(From a formal point of view, since we defined functions as set of pairs the two notions coincide: the restriction relation above is equivalent to requiring that $\langle a, b \rangle \in g \implies \langle a, b \rangle \in f$ for all a, b , which indeed states that g is a “subset” of f).

2 Question

Let \mathcal{F} be the set of partial functions $\{f \in (\mathbb{N} \rightsquigarrow \mathbb{N}) \mid \forall x \in \mathbb{N}. f(2 \cdot x) = x\}$.

- Define two distinct partial functions f_1, f_2 which belong to \mathcal{F} . (I.e, provide two such examples.)
- Define two distinct partial functions g_1, g_2 which do *not* belong to \mathcal{F} . (I.e, provide two such examples.)
- Define a partial function $f \in \mathcal{F}$, and consider the set of its *finite* restrictions $\mathcal{G} = \{g \in (\mathbb{N} \rightsquigarrow \mathbb{N}) \mid g \subseteq f \wedge \text{dom}(g) \text{ finite}\}$.
 - Define two distinct partial functions h_1, h_2 which belong to \mathcal{G} . (I.e, provide two such examples.)

- Prove whether $\mathcal{F} \cap \mathcal{G} = \emptyset$.

2.1 Answer

- $f_1(y) = \{y/2 \mid \forall k \in \mathbb{N}. y = 2k\}$, $f_2(y) = \{y^2/2y \mid \forall y \neq 0\}$
- $g_1(y) = \{y/3 \mid \forall k \in \mathbb{N}. y = 3k\}$, $g_2(y) = \{y/4 \mid \forall k \in \mathbb{N}. y = 4k\}$
- Choose $f(x) = \{x/2 \mid \forall k \in \mathbb{N}. x = 2k\}$, $h_1 = \{\frac{(5-2x)x}{x} \mid \forall x \in \text{dom}(h_1)\}$,
 $h_2 = \{\frac{(7-3x)x}{x} \mid \forall x \in \text{dom}(h_2)\}$
- We can find a function h_1 that satisfies $h_1 \in \mathcal{F} \wedge h_1 \in \mathcal{G}$. So $\mathcal{F} \cap \mathcal{G} \neq \emptyset$.

Note.

The next part is an advanced exercise. I'd suggest to **skip** it, unless you want an extra challenge.

3 Preliminaries

Let \mathcal{R} be a set of inference rules over elements of a set A . Then, \mathcal{R} induces a function $\hat{\mathcal{R}} \in (\mathcal{P}(A) \rightarrow \mathcal{P}(A))$ given by

$$\hat{\mathcal{R}}(X) = \{y \mid \exists(\frac{x_1 \dots x_n}{z}) \in \mathcal{R} \wedge y = z \wedge \forall i \in \{1, \dots, n\}. x_i \in X\}$$

4 Question

Let m, n range over natural numbers. Consider the following set of inference rules \mathcal{R}

$$\frac{n \ m}{n \cdot m} \quad \frac{}{1} \quad \frac{n}{n \cdot 2}$$

on the sets

$$E = \{2 \cdot n \mid n \in \mathbb{N}\} \quad O = \{2 \cdot n + 1 \mid n \in \mathbb{N}\}$$

Then, answer the questions below.

1. State whether $\hat{\mathcal{R}}(O) \subseteq O$
2. State whether $O \subseteq \hat{\mathcal{R}}(O)$
3. State whether $\hat{\mathcal{R}}(E) \subseteq E$
4. State whether $E \subseteq \hat{\mathcal{R}}(E)$
5. State whether $\hat{\mathcal{R}}(\mathbb{N}) \subseteq \mathbb{N}$

6. State whether $\mathbb{N} \subseteq \hat{\mathcal{R}}(\mathbb{N})$

7. State whether $\hat{\mathcal{R}}(E \cup \{1\}) \subseteq E \cup \{1\}$

You may wish to exploit the answer for some question when answering another.
Finally:

1. Characterize the minimum fixed point of $\hat{\mathcal{R}}$, i.e. $\bigcap \{X \mid \hat{\mathcal{R}}(X) = X\}$

2. Characterize the maximum fixed point of $\hat{\mathcal{R}}$, i.e. $\bigcup \{X \mid \hat{\mathcal{R}}(X) = X\}$

4.1 Answer

Write your answer here.