Computability Assignment
Year 2012/13 - Number 4

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1 Preliminaries

A partial function g is said to be a *restriction* of a partial function f , written $g \subseteq f$ iff

$$\forall x \in \mathsf{dom}(g).\ g(x) = f(x)$$

Note: this notation "overloads" the symbol $\subseteq$. Indeed, we shall write $A \subseteq B$ to express a subset relation between two sets, and $g \subseteq f$ to express a restriction relation between two functions.

(From a formal point of view, since we defined functions as set of pairs the two notions coincide: the restriction relation above is equivalent to requiring that $\langle a, b\rangle \in g \implies \langle a, b\rangle \in f$ for all a, b, which indeed states that g is a "subset" of f).

2 Question

Let $\mathcal{F}$ be the set of partial functions $\{f \in (\mathbb{N} \rightsquigarrow \mathbb{N}) | \forall x \in \mathbb{N}.\ f(2 \cdot x) = x\}$.

- Define two distinct partial functions f_1, f_2 which belong to $\mathcal{F}$. (I.e, provide two such examples.)
- Define two distinct partial functions g_1, g_2 which do *not* belong to $\mathcal{F}$. (I.e, provide two such examples.)
- Define a partial function $f \in \mathcal{F}$, and consider the set of its *finite* restrictions $\mathcal{G} = \{g \in (\mathbb{N} \rightsquigarrow \mathbb{N}) | g \subseteq f \ \wedge\ \mathsf{dom}(g) \text{ finite}\}$.
  - Define two distinct partial functions h_1, h_2 which belong to $\mathcal{G}$. (I.e, provide two such examples.)

- Prove whether $\mathcal{F} \cap \mathcal{G} = \emptyset$.

2.1 Answer

(RZ: the question was about partial functions $f \in (\mathbb{N} \rightsquigarrow \mathbb{N})$, not from A to B)

- $f_1 : A \to B$ where $A = \{0\} \wedge B = \{0\}$.
- $g_1 : A \to B$ where $A = \{2\} \wedge B = \{4\}$, g_2 the same with an other domain and range different from f_1.
- the examples of g_1 also applyes for h_1 and h_2.
- Let assume that $\mathcal{F} \cap \mathcal{G} = \emptyset$, take $g_1 \in \mathcal{F} \wedge h_1 \in \mathcal{G}$ as defined before. But $g_1 = h_1 \Rightarrow \mathcal{F} \cap \mathcal{G} \neq \emptyset$.

Note.

The next part is an advanced exercise. I'd suggest to **skip** it, unless you want an extra challenge.

3 Preliminaries

Let $\mathcal{R}$ be a set of inference rules over elements of a set A. Then, $\mathcal{R}$ induces a function $\hat{\mathcal{R}} \in (\mathcal{P}(A) \to \mathcal{P}(A))$ given by

$$\hat{\mathcal{R}}(X) = \{y \mid \exists(\frac{x_1 \ldots x_n}{z}) \in \mathcal{R} \wedge y = z \wedge \forall i \in \{1, \ldots, n\}.x_i \in X\}$$

4 Question

Let m, n range over natural numbers. Consider the following set of inference rules $\mathcal{R}$

$$\frac{n \ m}{n \cdot m} \qquad \frac{}{1} \qquad \frac{n}{n \cdot 2}$$

an the sets

$$E = \{2 \cdot n \mid n \in \mathbb{N}\} \qquad O = \{2 \cdot n + 1 \mid n \in \mathbb{N}\}$$

Then, answer the questions below.

1. State whether $\hat{\mathcal{R}}(O) \subseteq O$
2. State whether $O \subseteq \hat{\mathcal{R}}(O)$
3. State whether $\hat{\mathcal{R}}(E) \subseteq E$

4. State whether $E \subseteq \hat{\mathcal{R}}(E)$
5. State whether $\hat{\mathcal{R}}(\mathbb{N}) \subseteq \mathbb{N}$
6. State whether $\mathbb{N} \subseteq \hat{\mathcal{R}}(\mathbb{N})$
7. State whether $\hat{\mathcal{R}}(E \cup \{1\}) \subseteq E \cup \{1\}$

You may whish to exploit the answer for some question when answering another. Finally:

1. Characterize the minimum fixed point of $\hat{\mathcal{R}}$, i.e. $\bigcap\{X \mid \hat{\mathcal{R}}(X) = X\}$
2. Characterize the maximum fixed point of $\hat{\mathcal{R}}$, i.e. $\bigcup\{X \mid \hat{\mathcal{R}}(X) = X\}$

4.1 Answer

Following the numeration above:

1. No, because applying the second and the third roule you can always indroduce 2 in the set, which is not present in the set O.
2. No, because is not possible to generate 3 whit the three roules above.
3. No, because the second roule introduces 1.
4. No, because is not possible to generate 6.
5. Yes, because with the first and the second roule you can always decompose a number in its prime factors, so you can generate every number. (Scomposizione in fattori primi, non so se si traduca così)
6. Yes, with the same motivaion above.
7. Yes, because is not possible to generate odd numbers rather than 1 without addition.