Computability Assignment
Year 2012/13 - Number 4

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1 Preliminaries

A partial function g is said to be a *restriction* of a partial function f , written $g \subseteq f$ iff

$$\forall x \in \mathsf{dom}(g).\ g(x) = f(x)$$

Note: this notation "overloads" the symbol $\subseteq$. Indeed, we shall write $A \subseteq B$ to express a subset relation between two sets, and $g \subseteq f$ to express a restriction relation between two functions.

(From a formal point of view, since we defined functions as set of pairs the two notions coincide: the restriction relation above is equivalent to requiring that $\langle a, b\rangle \in g \implies \langle a, b\rangle \in f$ for all a, b, which indeed states that g is a "subset" of f).

2 Question

Let $\mathcal{F}$ be the set of partial functions $\{f \in (\mathbb{N} \rightsquigarrow \mathbb{N}) | \forall x \in \mathbb{N}.\ f(2 \cdot x) = x\}$.

- Define two distinct partial functions f_1, f_2 which belong to $\mathcal{F}$. (I.e, provide two such examples.)
- Define two distinct partial functions g_1, g_2 which do *not* belong to $\mathcal{F}$. (I.e, provide two such examples.)
- Define a partial function $f \in \mathcal{F}$, and consider the set of its *finite* restrictions $\mathcal{G} = \{g \in (\mathbb{N} \rightsquigarrow \mathbb{N}) | g \subseteq f\ \wedge\ \mathsf{dom}(g) \text{ finite}\}$.
  - Define two distinct partial functions h_1, h_2 which belong to $\mathcal{G}$. (I.e, provide two such examples.)

– Prove whether $\mathcal{F} \cap \mathcal{G} = \emptyset$.

2.1 Answer

- A partial function f_1 can be defined as

  $$f_1 = \begin{cases} \frac{x}{2} & if\ x\ is\ even \\ undef & ow \end{cases}$$

  (RZ: write $f_1(x)$, not f_1)

  and f_2 as:

  $$f_1 = \begin{cases} \frac{x}{2} & if\ x\ is\ even \\ \frac{x+1}{2} & ow \end{cases}$$

- g_1 and g_2 are not in F if for example doesen't accept the condition $f(2 \cdot x) = x$

  $g_2 = \frac{x}{2}$

  and

  $$g_2 = \begin{cases} x & x > 0 \\ \frac{x}{2} & x < 0 \end{cases}$$

  (RZ: how are these defined, exactly? $x < 0$ is always false on $\mathbb{N}$. What is $g_1(3)$? Undefined?)

- Now we can take f_1 and define over it the set of finite restriction $\mathcal{G} = \{g \in (\mathbb{N} \rightsquigarrow \mathbb{N}) | g \subseteq f \ \wedge \ \mathsf{dom}(g) \text{ finite}\}$

  this restriction means that we have to bound our function f_1 two example of this kind of bound can be:

  – $h_1 = \begin{cases} \frac{x}{2} & 0 < x < 10 \wedge x\ is\ even \\ undef & o.w. \end{cases}$

  – $h_2 = \begin{cases} \sqrt{x} & x = 4 \\ undef & o.w. \end{cases}$

  (RZ: ok)

- Now we can proof that $F \cap G = \emptyset$

  By contraddiction we take $F \cap G \neq \emptyset$ this means that must exist's at least a function such that have to be defined $\forall x \in \mathbb{N}$ (RZ: no, only on the even naturals) By definition of F so the $dom(f) = \mathbb{N}$ that is infinite and by definition of G $dom(f)$ has to be finite so in order to proof $F \cap G \neq \emptyset$ the domain of $\mathbb{N}$ has to be finite that's a contraddiction. So we have proved that $F \cap G = \emptyset$

Note

The next part is an advanced exercise. I'd suggest to **skip** it, unless you want an extra challenge.

3 Preliminaries

Let $\mathcal{R}$ be a set of inference rules over elements of a set A. Then, $\mathcal{R}$ induces a function $\hat{\mathcal{R}} \in (\mathcal{P}(A) \to \mathcal{P}(A))$ given by

$$\hat{\mathcal{R}}(X) = \{y \mid \exists(\frac{x_1 \ldots x_n}{z}) \in \mathcal{R} \wedge y = z \wedge \forall i \in \{1, \ldots, n\}.x_i \in X\}$$

4 Question

Let m, n range over natural numbers. Consider the following set of inference rules $\mathcal{R}$

$$\frac{n \; m}{n \cdot m} \qquad \frac{}{1} \qquad \frac{n}{n \cdot 2}$$

an the sets

$$E = \{2 \cdot n \mid n \in \mathbb{N}\} \qquad O = \{2 \cdot n + 1 \mid n \in \mathbb{N}\}$$

Then, answer the questions below.

1. State whether $\hat{\mathcal{R}}(O) \subseteq O$
2. State whether $O \subseteq \hat{\mathcal{R}}(O)$
3. State whether $\hat{\mathcal{R}}(E) \subseteq E$
4. State whether $E \subseteq \hat{\mathcal{R}}(E)$
5. State whether $\hat{\mathcal{R}}(\mathbb{N}) \subseteq \mathbb{N}$
6. State whether $\mathbb{N} \subseteq \hat{\mathcal{R}}(\mathbb{N})$
7. State whether $\hat{\mathcal{R}}(E \cup \{1\}) \subseteq E \cup \{1\}$

You may whish to exploit the answer for some question when answering another. Finally:

1. Characterize the minimum fixed point of $\hat{\mathcal{R}}$, i.e. $\bigcap\{X \mid \hat{\mathcal{R}}(X) = X\}$
2. Characterize the maximum fixed point of $\hat{\mathcal{R}}$, i.e. $\bigcup\{X \mid \hat{\mathcal{R}}(X) = X\}$

4.1 Answer

Write your answer here.