Computability Assignment
Year 2012/13 - Number 4

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Please do not submit a file containing only the answers; edit this file, instead, filling the answer sections.

1 Preliminaries

A partial function g is said to be a *restriction* of a partial function f, written $g \subseteq f$ iff

$$\forall x \in \mathsf{dom}(g).\ g(x) = f(x)$$

Note: this notation "overloads" the symbol $\subseteq$. Indeed, we shall write $A \subseteq B$ to express a subset relation between two sets, and $g \subseteq f$ to express a restriction relation between two functions.

(From a formal point of view, since we defined functions as set of pairs the two notions coincide: the restriction relation above is equivalent to requiring that $\langle a, b\rangle \in g \implies \langle a, b\rangle \in f$ for all a, b, which indeed states that g is a "subset" of f).

2 Question

Let $\mathcal{F}$ be the set of partial functions $\{f \in (\mathbb{N} \rightsquigarrow \mathbb{N}) | \forall x \in \mathbb{N}.\ f(2 \cdot x) = x\}$.

- Define two distinct partial functions f_1, f_2 which belong to $\mathcal{F}$. (I.e, provide two such examples.)
- Define two distinct partial functions g_1, g_2 which do *not* belong to $\mathcal{F}$. (I.e, provide two such examples.)
- Define a partial function $f \in \mathcal{F}$, and consider the set of its *finite* restrictions $\mathcal{G} = \{g \in (\mathbb{N} \rightsquigarrow \mathbb{N}) | g \subseteq f\ \wedge\ \mathsf{dom}(g) \text{ finite}\}$.
  - Define two distinct partial functions h_1, h_2 which belong to $\mathcal{G}$. (I.e, provide two such examples.)

– Prove whether $\mathcal{F} \cap \mathcal{G} = \emptyset$.

2.1 Answer

- $f_1(x) = \begin{cases} \frac{1}{2}x & x\ is\ even \\ undefined & otherwise \end{cases}$

  – $f_2(x) = \begin{cases} \frac{1}{2}x & x\ is\ even\ \wedge\ x < 10 \\ undefined & otherwise \end{cases}$ (The condition does not strictly hold here, because for some $x \in N$, $f(2x)$ is undefined) (RZ: so this proves $f_2 \notin \mathcal{F}$!! ;-))

- $g_1(x) = 0$
- $g_2(x) = 1/x$ (RZ: beware the result should be a natural)
- $f(x) = x$ (RZ: not in the set $\mathcal{F}$?)

  – $h_1(x) = \begin{cases} x & x < 10 \\ undefined & otherwise \end{cases}$

  – $h_2(x) = \begin{cases} x & x < 8 \\ undefined & otherwise \end{cases}$

  – If $f_2$is a correct answer, than it is an example of a function that is contained both in $\mathcal{F}$ and G, so the intersection is not empty.

Note.

The next part is an advanced exercise. I'd suggest to **skip** it, unless you want an extra challenge.

3 Preliminaries

Let $\mathcal{R}$ be a set of inference rules over elements of a set A. Then, $\mathcal{R}$ induces a function $\hat{\mathcal{R}} \in (\mathcal{P}(A) \to \mathcal{P}(A))$ given by

$$\hat{\mathcal{R}}(X) = \{y \mid \exists(\frac{x_1 \ldots x_n}{z}) \in \mathcal{R} \wedge y = z \wedge \forall i \in \{1, \ldots, n\}.x_i \in X\}$$

4 Question

Let m, n range over natural numbers. Consider the following set of inference rules $\mathcal{R}$

$$\frac{n \quad m}{n \cdot m} \qquad \frac{}{1} \qquad \frac{n}{n \cdot 2}$$

an the sets

$$E = \{2 \cdot n \mid n \in \mathbb{N}\} \qquad O = \{2 \cdot n + 1 \mid n \in \mathbb{N}\}$$

Then, answer the questions below.

1. State whether $\hat{\mathcal{R}}(O) \subseteq O$
2. State whether $O \subseteq \hat{\mathcal{R}}(O)$
3. State whether $\hat{\mathcal{R}}(E) \subseteq E$
4. State whether $E \subseteq \hat{\mathcal{R}}(E)$
5. State whether $\hat{\mathcal{R}}(\mathbb{N}) \subseteq \mathbb{N}$
6. State whether $\mathbb{N} \subseteq \hat{\mathcal{R}}(\mathbb{N})$
7. State whether $\hat{\mathcal{R}}(E \cup \{1\}) \subseteq E \cup \{1\}$

You may whish to exploit the answer for some question when answering another. Finally:

1. Characterize the minimum fixed point of $\hat{\mathcal{R}}$, i.e. $\bigcap\{X \mid \hat{\mathcal{R}}(X) = X\}$
2. Characterize the maximum fixed point of $\hat{\mathcal{R}}$, i.e. $\bigcup\{X \mid \hat{\mathcal{R}}(X) = X\}$

4.1 Answer

1. No, because $\hat{\mathcal{R}}(O)$ also includes even numbers
2. Yes, I can preserve all the odd numbers by multiplying them by 1 (which is odd)
3. No, because $\hat{\mathcal{R}}(E)$ also includes 1, which is not even.
4. No, because I cannot preserve the number 2.
5. Yes, because $\hat{\mathcal{R}}(\mathbb{N}) = \mathbb{N}$. In fact, we can multiply every number by 1 and still get the whole $\mathbb{N}$. The other numbers we obtain are still natural numbers.
6. Yes, see 5.
7. Yes, because $\hat{\mathcal{R}}(E \cup \{1\}) = E \cup \{1\}$. In fact, if we use rule 1, we still get $E \cup \{1\}$. With rule 2, we get 1 (which is already there), and with 3, we still get $E \cup \{1\}$.

Part 2:

1. Every X must be infinite, because otherwise rule #3 would generate a larger set (and it would be different). It must also contain 1, all the even numbers (zero is not required), so $E \cup \{1\} \setminus \{0\}$could be a possible answer.

2. $\mathbb{N}$(see 5. above)