Computability Assignment
Year 2012/13 - Number 4

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Please do not submit a file containing only the answers; edit this file, instead, filling the answer sections.

1   Preliminaries

A partial function g is said to be a *restriction* of a partial function f , written $g \subseteq f$ iff

$$\forall x \in \mathsf{dom}(g).\ g(x) = f(x)$$

Note: this notation "overloads" the symbol $\subseteq$. Indeed, we shall write $A \subseteq B$ to express a subset relation between two sets, and $g \subseteq f$ to express a restriction relation between two functions.

(From a formal point of view, since we defined functions as set of pairs the two notions coincide: the restriction relation above is equivalent to requiring that $\langle a, b \rangle \in g \implies \langle a, b \rangle \in f$ for all a, b, which indeed states that g is a "subset" of f).

2   Question

Let $\mathcal{F}$ be the set of partial functions $\{f \in (\mathbb{N} \rightsquigarrow \mathbb{N}) | \forall x \in \mathbb{N}.\ f(2 \cdot x) = x\}$.

- Define two distinct partial functions f_1, f_2 which belong to $\mathcal{F}$. (I.e, provide two such examples.)

- Define two distinct partial functions g_1, g_2 which do *not* belong to $\mathcal{F}$. (I.e, provide two such examples.)

- Define a partial function $f \in \mathcal{F}$, and consider the set of its *finite* restrictions $\mathcal{G} = \{g \in (\mathbb{N} \rightsquigarrow \mathbb{N}) | g \subseteq f\ \wedge\ \mathsf{dom}(g)$ finite$\}$.

    - Define two distinct partial functions h_1, h_2 which belong to $\mathcal{G}$. (I.e, provide two such examples.)

    – Prove whether $\mathcal{F} \cap \mathcal{G} = \emptyset$.

2.1 Answer

- $f_1(x) = \lfloor \frac{x}{2} \rfloor$, $f_2(x) = \lceil \frac{x}{2} \rceil$ (RZ: ok, nice)

- $g_1(x) = 2x$, $g_2(x) = x^2$

- $h_1(x) = \begin{cases} \frac{x}{2} & x \text{ is an even number within } 100 \\ \text{undefined} & \text{o.w.} \end{cases}$, $h_2(x) = \begin{cases} \sqrt{x} & x = 4 \\ \text{undefined} & \text{o.w.} \end{cases}$

- $\mathcal{F} \cap \mathcal{G} = \emptyset$. In $\mathcal{F}$, the domain of each function is infinite, as it has to include all the even numbers. While in $\mathcal{G}$, all functions are defined on a finite domain.

Note.

The next part is an advanced exercise. I'd suggest to **skip** it, unless you want an extra challenge.

3 Preliminaries

Let $\mathcal{R}$ be a set of inference rules over elements of a set A. Then, $\mathcal{R}$ induces a function $\hat{\mathcal{R}} \in (\mathcal{P}(A) \to \mathcal{P}(A))$ given by

$$\hat{\mathcal{R}}(X) = \{y \mid \exists (\frac{x_1 \ldots x_n}{z}) \in \mathcal{R} \land y = z \land \forall i \in \{1, \ldots, n\}.x_i \in X\}$$

4 Question

Let m, n range over natural numbers. Consider the following set of inference rules $\mathcal{R}$

$$\frac{n \; m}{n \cdot m} \qquad \overline{1} \qquad \frac{n}{n \cdot 2}$$

an the sets

$$E = \{2 \cdot n \mid n \in \mathbb{N}\} \qquad O = \{2 \cdot n + 1 \mid n \in \mathbb{N}\}$$

Then, answer the questions below.

1. State whether $\hat{\mathcal{R}}(O) \subseteq O$

2. State whether $O \subseteq \hat{\mathcal{R}}(O)$

3. State whether $\hat{\mathcal{R}}(E) \subseteq E$

4. State whether $E \subseteq \hat{\mathcal{R}}(E)$

5. State whether $\hat{\mathcal{R}}(\mathbb{N}) \subseteq \mathbb{N}$

6. State whether $\mathbb{N} \subseteq \hat{\mathcal{R}}(\mathbb{N})$

7. State whether $\hat{\mathcal{R}}(E \cup \{1\}) \subseteq E \cup \{1\}$

You may whish to exploit the answer for some question when answering another. Finally:

1. Characterize the minimum fixed point of $\hat{\mathcal{R}}$, i.e. $\bigcap\{X \mid \hat{\mathcal{R}}(X) = X\}$

2. Characterize the maximum fixed point of $\hat{\mathcal{R}}$, i.e. $\bigcup\{X \mid \hat{\mathcal{R}}(X) = X\}$

4.1 Answer

Consider $0 \in \mathbb{N}$.

1. No.$\hat{\mathcal{R}}(O) = \mathbb{N}$ (RZ: ok but $8 \notin \mathcal{R}\hat{(}O)$)

2. Yes.

3. No. $1 \in \hat{\mathcal{R}}(E)$

4. No. $2 \notin \hat{\mathcal{R}}(E)$

5. Yes.$\hat{\mathcal{R}}(\mathbb{N}) = \mathbb{N}$

6. Yes.

7. Yes.$\hat{\mathcal{R}}(E \cup \{1\}) = E \cup \{1\}$ (RZ: ok)

Fixed points,

1. Minimum fixed point: $E \cup \{1\}$. (RZ: not the minimum, try deriving 6 for instance)

2. Maximum fixed point:$\mathbb{N}$.