Computability Assignment
Year 2012/13 - Number 4

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Please do not submit a file containing only the answers; edit this file, instead, filling the answer sections.

1 Preliminaries

A partial function g is said to be a *restriction* of a partial function f , written $g \subseteq f$ iff

$$\forall x \in \mathsf{dom}(g).\ g(x) = f(x)$$

Note: this notation "overloads" the symbol $\subseteq$. Indeed, we shall write $A \subseteq B$ to express a subset relation between two sets, and $g \subseteq f$ to express a restriction relation between two functions.

(From a formal point of view, since we defined functions as set of pairs the two notions coincide: the restriction relation above is equivalent to requiring that $\langle a, b\rangle \in g \implies \langle a, b\rangle \in f$ for all a, b, which indeed states that g is a "subset" of f).

2 Question

Let $\mathcal{F}$ be the set of partial functions $\{f \in (\mathbb{N} \rightsquigarrow \mathbb{N}) | \forall x \in \mathbb{N}.\ f(2 \cdot x) = x\}$.

- Define two distinct partial functions f_1, f_2 which belong to $\mathcal{F}$. (I.e, provide two such examples.)
- Define two distinct partial functions g_1, g_2 which do *not* belong to $\mathcal{F}$. (I.e, provide two such examples.)
- Define a partial function $f \in \mathcal{F}$, and consider the set of its *finite* restrictions $\mathcal{G} = \{g \in (\mathbb{N} \rightsquigarrow \mathbb{N}) | g \subseteq f\ \wedge\ \mathsf{dom}(g) \text{ finite}\}$.
  - Define two distinct partial functions h_1, h_2 which belong to $\mathcal{G}$. (I.e, provide two such examples.)

– Prove whether $\mathcal{F} \cap \mathcal{G} = \emptyset$.

2.1 Answer

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NOTATION TRIVIA: the symbol $\uparrow$stands for "undefined" when talking about partial functions, and "diverges" (loops forever) when talking about programs.

NOTATION TRIVIA: similarly, the symbol $\downarrow$ stands for "converges" (outputs a result in a (possibly large) amount of finite time) when talking about programs.

NOTATION TRIVIA: the symbol $\perp$, pronounced "bottom", is borrowed from logic. It holds the constant value "false" and when found at the end of a formal proof highlightes a contradiction/absurd.

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a. $f_1(x) = \begin{cases} x/2 & if\ \exists n \in \mathbb{N}.x = 2n \\ \uparrow & else \end{cases}$, $f_2(x) = \begin{cases} x/2 & if\ \exists n \in \mathbb{N}.x = 2n \\ (x-1)/2 & else \end{cases}$

b. $g_1(x) = \begin{cases} (x-1)/2 & if\ \exists n \in \mathbb{N}.x = 2n+1 \\ \uparrow & else \end{cases}$, $g_2(x) = \begin{cases} x/2 & if\ \exists n \in \mathbb{N}.x = 2n \wedge n \leq ack(2,2,2) \\ \uparrow & else \end{cases}$

where g_1 and g_2 violate the requested behaviour on an infinite set of points. (ack is the ackermann function)

c. Let's fix $f = f_1$. Then:

$h_1(x) = \begin{cases} x/2 & if\ \exists n \in \mathbb{N}.1 \leq n \leq 32 \wedge x = 2^n \\ \uparrow & else \end{cases}$, $h_2(x) = \begin{cases} x/2 & if\ \exists n \in \mathbb{N}.n \leq 1024 \wedge x = 2^8 n \\ \uparrow & else \end{cases}$

d. Let's suppose for contraddiction that there exists f's.t. $f' \in (\mathcal{F} \cap \mathcal{G}) \neq \emptyset$.

$f' \in \mathcal{F} \implies |dom(f')| = \mathbb{N}$ (the set of even numbers has a bijection with $\mathbb{N}$)

$f' \in \mathcal{G} \implies |dom(f')| = k$ s.t. $k \in \mathbb{N}$ (by definition)

$\implies \perp$, hence $\mathcal{F} \cap \mathcal{G} = \emptyset$.

Note.

The next part is an advanced exercise. I'd suggest to **skip** it, unless you want an extra challenge.

3 Preliminaries

Let $\mathcal{R}$ be a set of inference rules over elements of a set A. Then, $\mathcal{R}$ induces a function $\hat{\mathcal{R}} \in (\mathcal{P}(A) \to \mathcal{P}(A))$ given by

$$\hat{\mathcal{R}}(X) = \{y \mid \exists(\frac{x_1 \dots x_n}{z}) \in \mathcal{R} \wedge y = z \wedge \forall i \in \{1, \dots, n\}.x_i \in X\}$$

4 Question

Let m, n range over natural numbers. Consider the following set of inference rules $\mathcal{R}$

$$\frac{n \; m}{n \cdot m} \qquad \frac{}{1} \qquad \frac{n}{n \cdot 2}$$

an the sets

$$E = \{2 \cdot n \mid n \in \mathbb{N}\} \qquad O = \{2 \cdot n + 1 \mid n \in \mathbb{N}\}$$

Then, answer the questions below.

1. State whether $\hat{\mathcal{R}}(O) \subseteq O$
2. State whether $O \subseteq \hat{\mathcal{R}}(O)$
3. State whether $\hat{\mathcal{R}}(E) \subseteq E$
4. State whether $E \subseteq \hat{\mathcal{R}}(E)$
5. State whether $\hat{\mathcal{R}}(\mathbb{N}) \subseteq \mathbb{N}$
6. State whether $\mathbb{N} \subseteq \hat{\mathcal{R}}(\mathbb{N})$
7. State whether $\hat{\mathcal{R}}(E \cup \{1\}) \subseteq E \cup \{1\}$

You may whish to exploit the answer for some question when answering another. Finally:

1. Characterize the minimum fixed point of $\hat{\mathcal{R}}$, i.e. $\bigcap\{X \mid \hat{\mathcal{R}}(X) = X\}$
2. Characterize the maximum fixed point of $\hat{\mathcal{R}}$, i.e. $\bigcup\{X \mid \hat{\mathcal{R}}(X) = X\}$

4.1 Answer

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1. No. The third inference rule injects even numbers into $\hat{\mathcal{R}}(O)$.

2. Yes. The first inference rule plus the fact that $1 \in O$ ensure that all odd numbers will remain into $\hat{\mathcal{R}}(O)$.

3. No. The second rule injects 1into $\hat{\mathcal{R}}(E)$, which does not belong to E itself.

4. No. The set E does not contain a couple of elements n, m s.t. $n \times m = 2(1 \notin E)$ nor an n s.t. $n \times 2 = 2$(same reason). Hence E contains an element that does not belong to $\hat{\mathcal{R}}(E)$.

5. Yes. The product operation (between natural numbers, Ed.) is closed over the set of natural numbers, and $1 \in \mathbb{N}$.

6. Yes. The first inference rule plus the fact that $1 \in \mathbb{N}$ ensure that all natural numbers will be into $\hat{\mathcal{R}}(\mathbb{N})$.

7. Yes. The second rule is not an problem anymore. The product of two even numbers is a another even number, the product with the neutral element returns the initial even number. The last rule is obviously not harmful at all.

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1. Minimum Fixed point is the set of powers of 2, P_2. Any $P' \subset P_2$ verifies $\hat{\mathcal{R}}(P') \neq P'$, while $\hat{\mathcal{R}}(P_2) = P_2$.

2. Maximum Fixed point is the set $\mathbb{N}$, because $\hat{\mathcal{R}}(\mathbb{N}) = \mathbb{N}$ so $\mathbb{N}$ belongs to the union.

[..was something more expected here?]

VYGER