Question 1. Let A, B be sets, and id_A, id_B are identity functions over A, B respectively. Assume $f \in (A \rightarrow B)$ and $g \in (B \rightarrow A)$ are functions satisfying $g \circ f = id_A$ and $f \circ g = id_B$. Prove that f is a bijection.

Solution 1. Suppose by contradiction f is either (i) not injective or (ii) not surjective.

Case (i), if f is not injective then there exists $a, a' \in A, a \neq a'$, with $f(a) = f(a')$. This implies that $|f(A)| \leq |A - 1|$, which implies that $|f(A)| < |A|$, now this implies that $|g(f(A))| < |A|$, since g is a function (and not a relation), and hence $g(f(A)) \neq A$, and hence our assumption that $g \circ f = id_A$ is a contradiction. (RZ: this is correct only if the sets are finite, with infinite sets $|A| - 1 = |A|$)

Case (ii), if f is not surjective then $f(A) \subset B$, hence $f(g(B)) \subset B$, hence our assumption that $f \circ g = id_B$, is a contradiction.

Since we proved that the necessary conditions for f not bieng bijective does not hold, f is bijective $\square$

Let A, B be sets, and let $f \in (A \leftrightarrow B)$ be a bijection. Define a bijection $g \in (P(A) \leftrightarrow P(B)$ and prove it is a bijection

Solution 2. Let $g : P(A) \rightarrow P(B)$ be such that, for any $A' \in P(A), g(A') = \{f(a) | a \in A'\}$, In order to prove that g is a bijection, we show that g is (i) injective and (ii) surjective

(i) By contradiction if g is not injective then there exists $A' \neq A'' \subseteq P(A)$, such that $g(A') = g(A'')$, which by definition implies that $f(A') = f(A'')$, but since $A' \neq A''$, our assumption that f is injective is a contradiction.

(ii) By contradiction if g is not surjective then there exists $B' \in P(B)$ such that, for any $A' \in P(A)$, $g(A') \neq B'$, but this implies that $f(A') \not\subseteq B$, but this is a contradiction to our assumption that $f \in (A \rightarrow B)$.

Since we proved that the necessary conditions for g not bieng bijective does not hold, g is bijective

Question 2. Let A and B be sets, and let $b \notin B$. Define a bijection f between the set of partial functions $(A \rightsquigarrow B)$ and the set of total functions $(A \rightarrow B \cup \{b\})$. Prove that it is a bijection

Solution 3. for any $g \in (A \rightsquigarrow B)$, define $f(g)$ as follows, for any $a \in A$,

$$f(g)(a) = \begin{cases} g(a) & \text{,if } g(a) \text{ is defined} \\ b & \text{,otherwise} \end{cases}$$

We prove that f is a bijection by proving (i) f is injective and (ii) surjective

(i) If f in injective, then for any $g, g' \in (A \rightsquigarrow B)$, if $g \neq g'$ then $f(g) \neq f(g')$. By contradiction, if $f(g) = f(g')$, then for any $(c, d) \in f(g)$, $(c, d) \in f(g')$ and vice versa. But this implies that $g = g'$, since by construction g and g' are respectively the sets obtained by removing the pairs (x, b), for any x, from $f(g)$ and $f(g'))$ respectively. Hence (i) is true

(ii) for any $g \in (A \to B \cup \{b\})$, we know from construction of f, $f^{-1}(g)$ is as follows, for any $a \in A$,

$$f^{-1}(g)(a) = \begin{cases} \text{undefined} & \text{, if } g(a) = b \\ g(a) & \text{, otherwise} \end{cases}$$

, since any such $f^{-1}(g) \in (A \rightsquigarrow B)$, f is surjective

Question 3. Define a bijection $f \in [(P(A) \times P(B)) \to P(A \uplus B)]$. Prove that is such

Solution 4. Let $1, 2$ be the tags assigned to the elements of sets A, B respectively, for the operation $\uplus$. Let f be defined as follows: for any $(A', B') \in P(A) \times P(B)$, as

$$f((A', B')) = \{(1, a) | a \in A'\} \cup \{(2, b) | b \in B'\}$$

In order to prove f is an bijection, we prove (i) f is injective and (ii) f is surjective.

(i) Suppose by contradiction, f is not injective then there exists pairs $(A', B'), (A'', B'') \in P(A) \times P(B)$, such that $(A', B') \neq (A'', B'')$ and $f((A', B')) = f((A'', B''))$. If $f((A', B')) = f((A'', B''))$, then, since by construction any element in $f((A', B'))$ is of the form $(1, a)$ or $(2, b)$, the set $\{a | (1, a) \in f((A', B'))\} = \{a | (1, a) \in f((A', B'))\}$, and $\{b | (2, b) \in f((A', B'))\} = \{b | (2, b) \in f((A'', B''))\}$. But this implies that $(A', B') = (A'', B'')$, which is a contradiction to our assumption.

(ii) In order to prove that f is surjective. By construction of f, the inverse of f, $f^{-1} : P(A \uplus B) \to P(A) \times P(B)$ is given as: For any set $X \in P(A \uplus B)$

$$f^{-1}(X) = \{(\{a | (1, a) \in X\}, \{b | (2, b) \in X\})\}$$

Now for any $X \in P(A \uplus B)$, one of the following mutually exclusive exhaustive cases is true :

1. X is empty – if this is the case, then $f^{-1}(X) = (\emptyset, \emptyset)$
2. $\{a | (1, a) \in X\}$ is not empty, and $\{b | (2, b) \in X\}$ is empty – if this is the case, then $f^{-1}(X)$ if of the form $(\{a | (1, a) \in X\}, \emptyset)$
3. $\{a | (1, a) \in X\}$ is empty, and $\{b | (2, b) \in X\}$ is not empty – if this is the case, then $f^{-1}(X) = (\emptyset, \{b | (2, b) \in X\})$
4. $\{a | (1, a) \in X\}$ is not empty, and $\{b | (2, b) \in X\}$ is not empty – if this is the case, then $f^{-1}(X) = (\{a | (1, a) \in X\}, \{b | (2, b) \in X\})$

Since in each of the above cases, show that f^{-1} is defined, f is surjective

Question 4. Define a bijection, $f \in [((A \uplus B) \to C) \leftrightarrow ((A \to C) \times (B \to C))]$

Solution 5. Let $1, 2$ be the tags assigned to the elements of sets A, B respectively, for the operation $\uplus$. We define the bijection f as follows: for any $g \in ((A \uplus B) \to C)$,

$$f(g) = (\{(a, c) | ((1, a), c) \in g\}, \{(b, c) | ((2, b), c) \in g\})$$

In order to prove that f is bijective, we prove f is both (i) injective and (ii) surjective.

(i) By contradiction, if f is not injective, then there exists $g, g' \in ((A \uplus B) \to C)$, $g \neq g'$ such that $f(g) = f(g')$. Let $f(g) = (h_1, h'_1)$ and $f(g') = (h_2, h'_2)$, but since $f(g) = f(g')$, $h_1 = h_2$ and $h'_1 = h'_2$. But by construction of f, this implies that $g = g'$.

(ii) In order to prove that f is surjective, We construct the inverse of f, $f^{-1} : ((A \to C) \times (B \to C)) \to ((A \uplus B) \to C)$ as: For any pair $(h, h') \in ((A \to B) \times (B \to C))$

$$f^{-1}((h, h')) = \{((1, a), c)|(a, c) \in h\} \cup \{((2, b), c)|(b, c) \in h'\}$$

Since f^{-1} is defined for any $(h, h') \in ((A \to B) \times (B \to C))$, f is surjective.

Question 5. Define a bijection, $f \in [(A \to (B \times C)) \leftrightarrow ((A \to B) \times (A \to C))]$

Solution 6. We define the bijection f as follows: for any $g \in (A \to (B \times C))$,

$$f(g) = \{((a, b), (a, c))|(a, (b, c)) \in g\}$$

In order to prove that f is bijective, we prove f is both (i) injective and (ii) surjective.

(i) By contradiction, if f is not injective, then there exists $g, g' \in (A \to (B \times C))$, $g \neq g'$ such that $f(g) = f(g')$. Let $f(g) = (h_1, h'_1)$ and $f(g') = (h_2, h'_2)$, but since $f(g) = f(g')$, $h_1 = h_2$ and $h'_1 = h'_2$. But by construction of f, this implies that $g = g'$.

(ii) In order to prove that f is surjective, We construct the inverse of f, $f^{-1} : ((A \to B) \times (A \to C)) \to (A \to (B \times C))$ as: For any pair $(h, h') \in ((A \to B) \times (B \to C))$

$$f^{-1}((h, h')) = \{(a, (b, c))|(a, b) \in h \wedge (a, c) \in h'\}$$

Since f^{-1} is defined for any $(h, h') \in ((A \to B) \times (A \to C))$, f is surjective.

[1]

[1] Sorry, I didn't manage to install the Lyx software on my computer yet, bcos there is a software upgrade error