Computability Assignment
Year 2012/13 - Number 3

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1 Question

Let A, B be sets, and let $\mathsf{id}_A, \mathsf{id}_B$ denote the identity functions over A and B respectively. Assume $f \in (A \to B)$ and $g \in (B \to A)$ be functions satisfying $g \circ f = \mathsf{id}_A$ and $f \circ g = \mathsf{id}_B$. Prove that f is a bijection (i.e., injective and surjective).

1.1 Answer

a. f is injective

let's prove it by contradiction

if it's not injective, then $\exists(x, y).\ f(x) = f(y) \nRightarrow x = y$

$\exists(x, y),\ x \neq y.\ f(x) = f(y)$

but if $f(x) = f(y)$ and $x \neq y$, then it cannot happen that $g(f(x)) = x$ and $g(f(y)) = y$, because the function f maps two different values of the domain to the same value of the codomain, and it is not possible to return to the distinct x and y by applying the function g

but this must happen, because $g \circ f = \mathsf{id}_A$

so it's impossible: f is injective

b. f is surjective

let's prove it by contradiction

if it's not surjective, then $\exists y \epsilon B.\ \forall x \epsilon A.\ f(x) \neq y$: there exists a y that is not mapped by g

but $f \circ g = f(g(y)) = \mathsf{id}_B = y$, so y must be mapped contradiction: f is surjective

f is injective and suriective, but it's bijective

2 Question

Let A, B be sets, and let $f \in (A \leftrightarrow B)$ be a bijection. Define a bijection $g \in (\mathcal{P}(A) \leftrightarrow \mathcal{P}(B))$ and prove it is such.

2.1 Answer

Let's take $A = \mathbb{N}$, $B = \mathbb{N} \setminus \{0\}$ and $f(x) = x + 1$. This is a bijection

let's define $\mathcal{P}(A) = \{0, 1, 2\}$, $\mathcal{P}(B) = \{2, 3, 4\}$, and $g(x) = x + 2$

P(A) and P(B) are subsets of A and B

g is a bijection, because it is a one-to-one mapping from $\mathcal{P}(A)$ to $\mathcal{P}(B)$:

$0 \to 2$

$1 \to 3$

$2 \to 4$

(RZ: ok, but in the general case?)

3 Question

Let A, B be two sets, and let $b \notin B$. Define a bijection f between the set of partial functions $(A \rightsquigarrow B)$ and the set of total functions $(A \to B \cup \{b\})$. Prove that it is such.

3.1 Answer

Let's take $A = \mathbb{N}$ and $B = \mathbb{N} \setminus \{0\}$. Let's take the set of partial functions C defined as $C = \{(c_1(1) = 2); (c_2(2) = 3); ...; (c_i(i) = i + 1); ...\}$, and the set of total functions $D : (\mathbb{N} \to \mathbb{N} \backslash \{0\} \cup \{0\})$, defined as $D = \{(d_1(x) = 1 \cdot x); (d_2(x) = 2 \cdot x); ...; (d_i(x) = i \cdot x); ...\}$ (where the c_is and the d_is are single functions) (RZ: what if they are different, in the general case?)

let's define the function $f : C \to D$ as $f(c_i) = d_i$

we have the following mapping:

$\triangleright \{1 \to 2\} \longrightarrow \{x \to x\}$

$\triangleright \{2 \to 3\} \longrightarrow \{x \to 2x\}$

$\triangleright ...$

$\triangleright \{i \to i + 1\} \longrightarrow \{x \to ix\}$

$\triangleright ...$

f is injective

if not, there exists a function d_i that is mapped by more than one different c_i

so $c_i \neq c_j \Rightarrow d_i = d_j$

so $\{i \rightarrow i+1\} \longrightarrow \{x \rightarrow ix\}$ and $\{j \rightarrow j+1\} \longrightarrow \{x \longrightarrow jx\}$, with $ix = jx$
but if $i \neq j$, than $ix \neq jx$ too
contradiction: f is injective

f is surjective
if not, there exists a function $d_z$that is not mapped by any c_i
d_i functions have the form $d_i(x) = ix$, so the d_z function should be $d_z(x) = zx$
but this, by definition, is mapped by the function $c_z(z) = z+1$
so f is surjective

Note.

The exercises below are harder. Feel free to skip them if you find them too hard.

4 Question

Define a bijection $f \in [(\mathcal{P}(A) \times \mathcal{P}(B)) \leftrightarrow \mathcal{P}(A \uplus B)]$. Prove that is is such.

4.1 Answer

Write your answer here.

5 Question

Define a bijection $f \in [((A \uplus B) \rightarrow C) \leftrightarrow ((A \rightarrow C) \times (B \rightarrow C))]$. Prove that is is such.

5.1 Answer

Write your answer here.

6 Question

Define a bijection $f \in [((A \rightarrow (B \times C)) \leftrightarrow ((A \rightarrow B) \times (A \rightarrow C))]$. Prove that is is such.

6.1 Answer

Write your answer here.