Computability Assignment
Year 2012/13 - Number 3

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1 Question

Let A, B be sets, and let $\mathsf{id}_A, \mathsf{id}_B$ denote the identity functions over A and B respectively. Assume $f \in (A \to B)$ and $g \in (B \to A)$ be functions satisfying $g \circ f = \mathsf{id}_A$ and $f \circ g = \mathsf{id}_B$. Prove that f is a bijection (i.e., injective and surjective).

1.1 Answer

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TRIVIA: $f \circ g \in (B \to B), (f \circ g)(b) = f(g(b)).\forall b \in B$, $g \circ f \in (A \to A), (g \circ f)(a) = g(f(a)).\forall a \in A$

TRIVIA: $id_B \in (B \to B), id_B(b) = b.\forall b \in B$, $id_A \in (A \to A), id_A(a) = a.\forall a \in A$

DEMONSTRATION:

1. Injectivity

By definition, a (total) function is a relation that maps an input of the domain in exactly one point of the codomain.

Let's suppose for contraddiction that there exist two elements a_1, a_2 such that $b = f(a_1)$ and $b = f(a_2)$ with $a_1 \neq a_2$.

Since $g \circ f = \mathsf{id}_A$, we have that $g(f(a_1)) = g(b) = a_1$ and $g(f(a_2)) = g(b) = a_2$.

However, g is a (total) function therefore there must exist only one element $a \in A$ such that $g(b) = a$.

Hence $a_1 = a_2$.

2. Surjectivity

Let's suppose for contraddiction that $\exists b \in B.\forall a \in A.f(a) \neq b$.

Then for this element b we have that $f \circ g(b) \neq b$, otherwise it would exist an $a \in A$ such that $id_B(b) = f \circ g(b) = f(g(b)) = f(a) = b$. $\perp$ (this is exactly one of our preliminary hyphotesis)

Hence, f is surjective (same holds for g)

3. Injective && Surjective => Bijective.

2 Question

Let A, B be sets, and let $f \in (A \leftrightarrow B)$ be a bijection. Define a bijection $g \in (\mathcal{P}(A) \leftrightarrow \mathcal{P}(B))$ and prove it is such.

2.1 Answer

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$g : \mathcal{P}(A) \leftrightarrow \mathcal{P}(B), g = \{(A', B') | A' \in \mathcal{P}(A) \wedge B' \in \mathcal{P}(B) \wedge B' = \bigcup_{a' \in A'} f(a')\}$.

(RZ: maybe $g = \{(A', B') | A' \in \mathcal{P}(A) \wedge B' \in \mathcal{P}(B) \wedge B' = \{f(a') \mid a' \in A'\}\}$ is a more clear notation)

1. Injectivity

Let's suppose for contraddiction that there exist two elements $A_1, A_2 \in \mathcal{P}(A)$ such that $B' = g(A_1)$ and $B' = g(A_2)$ ($B' \in \mathcal{P}(B)$) with $A_1 \neq A_2$.

$A_1 \neq A_2 \implies \exists a \in (A_1 \cup A_2) \backslash (A_1 \cap A_2) \implies \exists b \in B. b = f(a)$. (since $a \in A$)

Without loss of generality, let's suppose $a \in A_1$.

Then by definition of g, $(A_1, B') \in g \implies b \in B'$ and also $b \in B' \wedge (A_2, B') \in g \implies a = f^{-1}(b) \in A_2$, contraddiction.

Hence, g is injective.

2. Surjectivity

Let's suppose for contraddiction that $\exists B' \in \mathcal{P}(B). \forall A' \in \mathcal{P}(A). g(A') \neq B'$.

Since f is a bijection, it has an inverse $f^{-1} \in (B \leftrightarrow A)$. So, let's define a new set parametric in B':

$A_{B'} = \{a \in A | \exists b \in B'. a = f^{-1}(b)\}$. It's easy to see that $A_{B'} \subseteq A$ (hence $A_{B'} \in \mathcal{P}(A)$), and that $B' = \bigcup_{a \in A_{B'}} f(a)$.

But this means that $(A_{B'}, B') \in g$, since we have fulfilled all the requirements. Contraddiction.

Hence, g is surjective.

3. Injective && Surjective => Bijective

3 Question

Let A, B be two sets, and let $b \notin B$. Define a bijection f between the set of partial functions $(A \rightsquigarrow B)$ and the set of total functions $(A \to B \cup \{b\})$. Prove that is is such.

3.1 Answer

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Since I've a narrowed imagination, i want to verify in advance that such a bijection might exist.

THEOREM: Among two finite sets there exists a bijection if and only if they have the same cardinality.

So let's do some preliminary considerations, in case A, B are finite sets:

a. The set of total functions $(A \rightarrow B)$ has cardinality$|(A \rightarrow B)| = |B|^{|A|}$, with no restrictions of type "surjectivity/injectivity".

b. The set of partial functions $(A \rightsquigarrow B)$ has cardinality $|(A \rightsquigarrow B)| = 1+\sum_{i=1}^{|A|} |B|^i \frac{|A|!}{(|A|-i)!i!}$, with no restrictions of type "surjectivity/injectivity". [first "1" accounts for the empty function]

c. The BINOMIAL FORMULA states $(1+x)^n = \sum_{i=0}^{n} x^i \frac{n!}{(n-i)!i!}$, therefore..

d. ..the set of total functions $(A \rightarrow B \cup \{b\})$, $b \notin B$, has cardinality $|(A \rightarrow B \cup \{b\})| = (1+|B|)^{|A|} = \sum_{i=0}^{|A|} |B|^i \frac{|A|!}{(|A|-i)!i!} = 1 + \sum_{i=1}^{|A|} |B|^i \frac{|A|!}{(|A|-i)!i!}$

Note: I've calculated these formulas without the support of references, so I advice you to be suspicious and verify them.

Trusting my estimations, one can conclude that such a bijection exists in the case of finite sets.

That's great,. With this renewed selfconfidence let's try to think harder to the (obvious) example.

A possible bijection could be $f = \{(h,g)|h : (A \rightsquigarrow B) \wedge g : (A \rightarrow B \cup \{b\}) \wedge h = \{(x,y)|x \in A \wedge y \in B \wedge (x,y) \in g\}\}$, $f : ((A \rightsquigarrow B) \leftrightarrow (A \rightarrow B \cup \{b\}))$.

Intuitively, two functions $h : (A \rightsquigarrow B), g : (A \rightarrow B \cup \{b\})$ are in relation by means of f iff the partial function h is the restriction of g on $(A \rightsquigarrow B)$.

So let's try to demonstrate that f is bijective:

1. Injectivity

Let $h_1, h_2 \in (A \rightsquigarrow B)$ be s.t. $f(h_1) = f(h_2) = g \in (A \rightarrow B \cup \{b\})$, we want to show that $h_1 = h_2$ follows necessarily.

$g = f(h_1) = f(h_2) \implies (h_1, g) \in f \wedge (h_2, g) \in f \implies h_1 = h_2 = \{(x,y)|x \in A \wedge y \in B \wedge (x,y) \in g\}$

2. Surjectivity

Let's take an arbitrary $g \in (A \rightarrow B \cup \{b\})$, and build its own restriction $z_g = \{(x,y)|x \in A \wedge y \in B \wedge (x,y) \in g\}$.

By definition of z_g , $dom(z_g) \subseteq A$ and $range(z_g) \subseteq B$, so it is a partial function belonging to $(A \rightsquigarrow B)$.

3. Totality

It is important to remark that f is total over the set $(A \rightsquigarrow B)$, and this follows intuitively by the observation that one can build

a total function $g : (A \rightarrow B \cup \{b\})$ starting from $h : (A \rightsquigarrow B)$ using the following definition

$g_h = \{(x,y)|x \in A \wedge y \in B \wedge ((x,y) \in h \vee y = b)\}$

which univocally associates a function g_h to each h. You can think of $\{b\}$ as a marker that states wherever h is undefined.

3. Injectivity && Surjectivity => Bijectivity

To conclude, let's observe that the definition of f is absolutely general and makes no assumptions on the cardinality

of the sets. Therefore it can be used also for infinite sets.

Note.

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The exercises below are harder. Feel free to skip them if you find them too hard.

4 Question

Define a bijection $f \in [(\mathcal{P}(A) \times \mathcal{P}(B)) \leftrightarrow \mathcal{P}(A \uplus B)]$. Prove that is is such.

4.1 Answer

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Again, we may repeat some cardinality considerations for the finite sets case:

a. $|(\mathcal{P}(A) \times \mathcal{P}(B)| = |\mathcal{P}(A)| \times |\mathcal{P}(B)| = 2^{|A|} + 2^{|B|} = 2^{|A|+|B|}$

b. $|\mathcal{P}(A \uplus B)| = 2^{|A \uplus B|} = 2^{|A|+|B|}$

the cardinality is equal in both cases, therefore there exists a bijection among the two sets.

TRIVIA: if A is a set, then $A \times \emptyset = \emptyset$.

So let's try to define one, assuming the existence of two arbitrary (possibly infinite) sets A, B:

$f_{A,B} = \{(X, Y)|(X = (A', B') \land A' \subseteq A \land B' \subseteq B) \implies Y = (\bigcup_{a \in A'} < a, 0 >) \cup (\bigcup_{b \in B'} < b, 1 >)\}$, $f : (\mathcal{P}(A) \times \mathcal{P}(B)) \leftrightarrow \mathcal{P}(A \uplus B)$

1. Injectivity

Suppose $X_1, X_2 \in (\mathcal{P}(A) \times \mathcal{P}(B))$ and $Y = f_{A,B}(X_1) = f_{A,B}(X_2) \in \mathcal{P}(A \uplus B)$.

$f_{A,B}(X_1) = f_{A,B}(X_2)$

$\implies X_1 = (A', B') \land X_2 = (A", B") \land Y = (\bigcup_{a \in A'} < a, 0 >) \cup (\bigcup_{b \in B'} < b, 1 >) = (\bigcup_{a \in A"} < a, 0 >) \cup (\bigcup_{b \in B"} < b, 1 >)$

$\implies (\bigcup_{a \in A'} < a, 0 >) = (\bigcup_{a \in A"} < a, 0 >) \land (\bigcup_{b \in B'} < b, 1 >) = (\bigcup_{b \in B"} < b, 1 >)$ [I left behind some obvious steps that lead to the next one]

$\implies A' = A" \land B' = B" \implies X_1 = (A', B') = (A", B") = X_2$

2. Surjectivity

Let's take an arbitrary $Y \in \mathcal{P}(A \uplus B)$ and construct $X = \{(A', B')|A' = (\bigcup_{<a,0> \in Y} a) \land B' = (\bigcup_{<b,1> \in Y} b)\}$.

It's easy to see that $A' \subseteq A$ and $B' \subseteq B$, therefore $X \in (\mathcal{P}(A) \times \mathcal{P}(B))$ and $f_{A,B}(X) = Y$.

$\implies f_{A,B}$ is injective, surjective and bijective.

5 Question

Define a bijection $f \in [((A \uplus B) \to C) \leftrightarrow ((A \to C) \times (B \to C))]$. Prove that is is such.

5.1 Answer

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Again, let's try to repeat some cardinality analisys in the case of finite sets.

a. $|((A \uplus B) \to C)| = |C|^{|(A \uplus B)|} = |C|^{(|A|+|B|)}$

b. $|(A \to C) \times (B \to C)| = |(A \to C)| \times |(B \to C)| = |C|^{|A|} \times |C|^{|B|} = |C|^{(|A|+|B|)}$

since the cardinalities coincide, we may conclude that such a bijection exists (at least) in the case of finite sets.

Good. Let A, B, C be arbitrary (possibly infinite) sets, and $g : ((A \uplus B) \to C)$, $h : (A \to C)$, $r : (B \to C)$ be placeholders for any function of that type.

Then a bijective $f \in [((A \uplus B) \to C) \leftrightarrow ((A \to C) \times (B \to C))]$ can be defined as:

$f = \{(g, (h, r)) | h = \{(a, c) | (< a, 0 >, c) \in g\} \wedge r = \{(b, c) | (< b, 1 >, c) \in g\}\}$

1. Injective

Let $g_1, g_2 \in ((A \uplus B) \to C)$ and

$(h, r) = f(g_1) = f(g_2) \in ((A \to C) \times (B \to C))$

$\implies (h = \{(a, c) | (< a, 0 >, c) \in g_1\} = \{(a, c) | (< a, 0 >, c) \in g_2\}) \wedge (r = \{(b, c) | (< b, 1 >, c) \in g_1\} = \{(b, c) | (< b, 1 >, c) \in g_2\})$

$\implies (\neg \exists (a, c) \in h.(< a, 0 >, c) \in (g_1 \cup g_2) \setminus (g_1 \cap g_2)) \wedge (\neg \exists (b, c) \in r.(< b, 1 >, c) \in (g_1 \cup g_2) \setminus (g_1 \cap g_2))$

$\implies g_1 = g_2$

2. Surjective

Pick an arbitrary $(h, r) \in ((A \to C) \times (B \to C))$, it's possible to define a function

$g_{(h,r)} = \{(x, c) | (x =< a, 0 > \wedge (a, c) \in h) \vee (x =< b, 1 > \wedge (b, c) \in r)\}$

It's easy to see, by the properties of functions h, r, that $dom(g_{(h,r)}) = (A \uplus B)$ and $range(g_{(h,r)}) \subseteq C$, $\implies g_{(h,r)} \in ((A \uplus B) \to C)$.

By construction, $f(g_{(h,r)}) = (h, r)$. Given that (h, r) has been chosen arbitrarily, we can conclude that f is surjective.

$\implies f$ is injective, surjective and bijective.

6 Question

Define a bijection $f \in [((A \to (B \times C)) \leftrightarrow ((A \to B) \times (A \to C))]$. Prove that is is such.

6.1 Answer

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Cardinality Analisys, for finite sets case:

a. $|(A \to (B \times C))| = |(B \times C)|^{|A|} = (|B| \times |C|)^{|A|}$

b. $|(A \to B) \times (A \to C)| = |(A \to B)| \times |(A \to C)| = |B|^{|A|} \times |C|^{|A|} = (|B| \times |C|)^{|A|}$

since the cardinalities coincide, we may conclude that such a bijection exists (at least) in the case of finite sets.

Let A, B, C be arbitrarily chosen (possibly infinite) sets, and $g : (A \to (B \times C))$, $h : (A \to C)$, $r : (A \to C)$ be placeholders for any function of the specified type.

Then a bijective $f \in [((A \to (B \times C)) \leftrightarrow ((A \to B) \times (A \to C))]$ can be defined as:

$f = \{(g, (h, r)) | h = \bigcup_{(a,(b,c)) \in g} (a, b) \wedge r = \bigcup_{(a,(b,c)) \in g} (a, c)\}$

1. Injective

Let $g_1, g_2 \in (A \to (B \times C))$ and

$(h, r) = f(g_1) = f(g_2) \in ((A \to B) \times (A \to C))$

$\implies (h = \bigcup_{(a,(b,c')) \in g_1} (a, b) = \bigcup_{(a,(b,c")) \in g_2} (a, b)) \wedge (r = \bigcup_{(a,(b',c)) \in g_1} (a, c) = \bigcup_{(a,(b",c)) \in g_2} (a, c))$

Pick an arbitrary $(a, (b, c)) \in g_1$, we want to show that it also belongs to g_2.

$(a, (b, c)) \in g_1 \wedge f(g_1) = f(g_2) \implies \exists (a, (b, c')) \in g_2 \wedge \exists (a, (b', c)) \in g_2$

since $g_2$is a function, we know that $g_2(a) = (b, c') = (b', c) \implies b = b' \wedge c = c'$

Hence $(a, (b, c)) \in g_2$ and $g_1 \subseteq g_2$. Since the proof can be done backwards too, $g_1 = g_2$.

2. Surjective

Pick an arbitrary $(h, r) \in ((A \to B) \times (A \to C))$ and construct a function

$g_{(h,r)} = \{(a, (b, c)) | b = h(a) \wedge c = r(a)\}$

then, by properties of functions h, r, $dom(g_{(h,r)}) = A$ and $range(g_{(h,r)}) \subseteq (B \times C)$ therefore $g_{(h,r)} \in ((A \to (B \times C))$.

By construction, $f(g_{(h,r)}) = (h, r)$. Given that (h, r) has been chosen arbitrarily, we can conclude that f is surjective.

$\implies f$ is injective, surjective and bijective.

Vyger