Computability Assignment
Year 2012/13 - Number 3

Please keep this file anonymous: do not write your name inside this file.

More information about assignments at http://disi.unitn.it/∼zunino/teaching/computability/assignments

Please do not submit a file containing only the answers; edit this file, instead, filling the answer sections.

1 Question

Let A, B be sets, and let $\mathsf{id}_A, \mathsf{id}_B$ denote the identity functions over A and B respectively. Assume $f \in (A \to B)$ and $g \in (B \to A)$ be functions satisfying $g \circ f = \mathsf{id}_A$ and $f \circ g = \mathsf{id}_B$. Prove that f is a bijection (i.e., injective and surjective).

1.1 Answer

By contradiction. (injective) Suppose f is not injective, $\exists a_1, a_2 \in A.\ a_1 \neq a_2.\ f(a_1) = f(a_2) = b \in B$, and we have $g \circ f(a_1) = g \circ f(a_2) = g(b)$, but since the composite function is an identity function, we also have $g \circ f(a_1) = a_1 \neq g \circ f(a_2) = a_2$, this is a contradiction.

(surjective) Suppose f is not surjective, $\exists b \in B. \nexists a \in A. f(a) = b$, but since $f \circ g(b) = b$, i.e.$\exists a \in A. g(b) = a \wedge f(a) = b$, such a exists.

2 Question

Let A, B be sets, and let $f \in (A \leftrightarrow B)$ be a bijection. Define a bijection $g \in (\mathcal{P}(A) \leftrightarrow \mathcal{P}(B))$ and prove it is such.

2.1 Answer

We define $g\ : X \to Y,\ Y = \{f(a) | a \in X\}$.(RZ: $g \in (\mathcal{P}(A) \leftrightarrow \mathcal{P}(B)),\ g(X) = \{f(a) | a \in X\}$)

By contradiction. (injective) Say $\exists X_1 \neq X_2 \in \mathcal{P}(A)$,$g(X_1) = g(X_2)$, but since f is a bijection, $g(X_1) \neq g(X_2)$.

(RZ: OK, but why?)

(surjective) Say $\exists Y \in \mathcal{P}(B).\nexists X \in \mathcal{P}(A).g(X) = Y$, but we can build such a $X = \{a | f(a) = y \wedge y \in Y\} \subseteq A$.

3 Question

Let A, B be two sets, and let $b \notin B$. Define a bijection f between the set of partial functions $(A \rightsquigarrow B)$ and the set of total functions $(A \to B \cup \{b\})$. Prove that it is such.

3.1 Answer

A partial function in $(A \rightsquigarrow B)$ can be written as

$$p(x) = \begin{cases} y & y \in B \\ \text{undefined} & \end{cases}$$

(RZ: this notation is a bit misleading)

We keep the mapping of the defined ones. We map all the rest on to b to make it total.

$$f(p)(x) = \begin{cases} p(x) & x \text{is defined on } p \\ b & x \text{is undefined on } p \end{cases}$$

We prove it is a bijection. (injective) Two partial functions differ only on the defined parts, by the above construction, they remain different. i.e.$\nexists p_1 \neq p_2 . f(p_1) = f(p_2)$. (RZ: a bit more detail?)

(surjective) We can always find a pre-image for a total function in $(A \to B \cup \{b\})$ by let those mapped to b be undefined. In the case of $g : A \to B$, the pre-image of g is g itself, which is a special case of partial functions.

Note.

The exercises below are harder. Feel free to skip them if you find them too hard.

4 Question

Define a bijection $f \in [(\mathcal{P}(A) \times \mathcal{P}(B)) \leftrightarrow \mathcal{P}(A \uplus B)]$. Prove that it is such.

4.1 Answer

Consider $S_A \in \mathcal{P}(A)$, $S_B \in \mathcal{P}(B)$, we define f as

$$f(\langle S_A, S_B \rangle) = \{\langle 0, a \rangle | a \in S_A\} \cup \{\langle 1, b \rangle | b \in S_B\}$$

f is injective. $\forall \langle S_A, S_B\rangle \neq \langle T_A, T_B\rangle$, we have $f(\langle S_A, S_B\rangle) \neq f(\langle T_A, T_B\rangle)$ since they are disjointed, and the difference remains.

f is surjective. $\forall S_{A\uplus B} \in \mathcal{P}(A \uplus B)$, we can build$S_A = \{a | \langle 0, a\rangle \in S_{A\uplus B}\}$,$S_B = \{b | \langle 1, b\rangle \in S_{A\uplus B}\}$, and$\langle S_A, S_B\rangle \in \mathcal{P}(A) \times \mathcal{P}(B)$.

5 Question

Define a bijection $f \in [((A \uplus B) \to C) \leftrightarrow ((A \to C) \times (B \to C))]$. Prove that it is such.

5.1 Answer

Consider $g \in ((A \uplus B) \to C),\ h \in (A \to C),\ j \in (B \to C)$. We define such an f.

$$f(g) = (h,\ j) \text{ where } h(a) = c \text{ iff } g(\langle 0, a\rangle) = c,\ j(b) = c' \text{ iff } g(\langle 1, b\rangle) = c'$$

f is injective. Assume $f(g_1) = f(g_2) = (h, j)$, then $\forall a \in A.g_1(\langle 0, a\rangle) = g_2(\langle 0, a\rangle) = h(a)$, the same applies for $\forall b \in B$, which suggests $g_1 = g_2$.

f is surjective. By the construction of f, we can always build a g for any $(h,\ j)$.

6 Question

Define a bijection $f \in [((A \to (B \times C)) \leftrightarrow ((A \to B) \times (A \to C))]$. Prove that it is such.

6.1 Answer

Consider $g \in (A \to (B \times C),\ h \in (A \to B),\ j \in (A \to C)$, we define such an f.

$$f(g) = (h,\ j) \text{ where } h(a) = b,\ j(a) = c \text{ iff } g(a) = (b,\ c)$$

We show f is injective. Assume $f(g_1) = f(g_2) = (h,\ j)$, then $\forall a \in A.g_1(a) = g_2(a) = (h(a),\ j(a))$, which means $g_1 = g_2$.

(Surjective) Similarly, we can always build a g for any $(h,\ j)$.