Computability Assignment
Year 2012/13 - Number 2

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1 Question

Let A, B be two sets. Prove that the properties below are equivalent.

- $A = \emptyset \vee B = \emptyset$
- $A \times B = \emptyset$

1.1 Answer

Since we have A x B $= \{< x, y > | x \in A \bigwedge y \in B\} \Rightarrow if\, A\, x\, B = \emptyset \iff A = \emptyset \vee\, B = \emptyset or \iff A = \emptyset\, \wedge B\, = \emptyset$

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i \in \mathbb{N}}$, we define $\bigcup_{i=0}^{\infty} A_i = \bigcup\{A_i \mid i \in \mathbb{N}\}$ and $\bigcup_{i=0}^{k} A_i = \bigcup\{A_i \mid i \in \mathbb{N} \ \wedge \ i \leq k\} = A_0 \cup A_1 \cup \cdots \cup A_k$.

3 Question

Assume $(A_i)_{i \in \mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \cdots \subseteq \mathbb{N} \ (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or

- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcup_{i=0}^{k} A_i$
2. p_2: for all i, if A_i is infinite, then $A_i = A_{i+1}$
3. p_3: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$
4. p_4: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is finite
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite

3.1 Answer

1. (*) is sufficient to say that p1 holds. Proof:
   $\forall k \in \mathbb{N}.\ A_k = \bigcup_{i=0}^{k} A_i\, if\, k = 0\ \Rightarrow A_0 = A_0$
   $if\, k = n\, from\, (*)\, we\, have\, A_n = A_{1\cup\ldots\cup}A_n\, but\, for\, definition\, the\, Set\, A_n\, contains\, every\, previous\, set.$
   $The\, equlity\, A_n = A_n\, holds,\, that\, implies\, A_k = \bigcup_{i=0}^{k} A_i$
2. (*) is sufficient to say that p2 doesn't hold. From (*) we have
   $A_i \subseteq \mathbb{N}\, A_{i+1} \subseteq \mathbb{N}. If\, we\, choosen\, the\, case\, with\, A_i\backslash\{0\}\, and\, A_{i+1}\backslash\{1\}\, the\, property\, p2\, is\, false.$
3. (*) is sufficient to say that p3 doesn't hold.
   $A_i \neq A_{i+1}\ \Rightarrow A_i \subset A_{i+1}\, from\, (*)$
   $but\, if\, we\, choosen\, A_1 \setminus \{0\}\ \Rightarrow \bigcup_{i=0}^{\infty} A_i \neq \mathbb{N}$
4. (*) Is not sufficient because we can have
   $\forall i\ \in \mathbb{N}\, A_i \subset A_{i+1} \Rightarrow \bigcup_{i=0}^{\infty} A_i\, is\, not\, finite.$
   $otherwise\, if\, we\, have\, \forall i\ \in \mathbb{N}\, A_i = A_{i+1}\ \Rightarrow \bigcup_{i=0}^{\infty} A_i\, is\, finite.$