Computability Assignment
Year 2012/13 - Number 2

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1 Question

Let A, B be two sets. Prove that the properties below are equivalent.

- $A = \emptyset \vee B = \emptyset$
- $A \times B = \emptyset$

1.1 Answer

If $A = \emptyset$: there is no element in A to create $< a, b > \in A \times B$, therefore $A \times B$ is empty; the same goes if $B = \emptyset$.

If $A \times B = \emptyset$: there is no couple $< a, b >$in the cross product, either the first element a in missing (meaning $A = \emptyset$) or the second element b is (meaning $B = \emptyset$). They both could be missing, of course.

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i\in\mathbb{N}}$, we define $\bigcup_{i=0}^{\infty} A_i = \bigcup\{A_i \mid i \in \mathbb{N}\}$ and $\bigcup_{i=0}^{k} A_i = \bigcup\{A_i \mid i \in \mathbb{N} \wedge i \leq k\} = A_0 \cup A_1 \cup \cdots \cup A_k$.

3 Question

Assume $(A_i)_{i\in\mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \cdots \subseteq \mathbb{N} \ (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcup_{i=0}^{k} A_i$
2. p_2: for all i, if A_i is infinite, then $A_i = A_{i+1}$
3. p_3: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$
4. p_4: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is finite
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite

3.1 Answer

Write your answer here.

1. $(*)$ states that $A_i \subseteq A_{i+1} \forall i \in \mathbb{N}$, so $A_{k-1} \equiv \bigcup_{i=0}^{k-1} A_i \subseteq A_k$ and $A_{k-1} \cup A_k \equiv A_k$, so $p_1$holds;
2. even if $|A_i| = \infty$, it could be that $A_i \neq A_{i+1}$; for example, $A_i \equiv \{x | x \text{ is even}\}$, $A_{i+1} \equiv \{x | x \text{ is even} \vee x = 1\}$, they both are infinite and $A_i \subset A_{i+1}$but $A_i \neq A_{i+1}$:$p_2$doesn't hold;
3. $\forall i \in \mathbb{N}. A_{i+1} \neq A_i$ but also (thanks to$(*)$) $A_i \subseteq A_{i+1}$, so $\exists a \in A_{i+1}. a \notin A_i$. Every set is bigger than its predecessor, so eventually $A_i = \mathbb{N}$ (for some i) and the union of them all is equal to $\mathbb{N}$, so p_3 holds;
4. for 1., $A_k = \bigcup_{i=0}^{k} A_i$. If A_i is finite $\forall i \in \mathbb{N}$, then A_k also is finite, so$p_4$holds;
5. contraddiction with 4., $p_5$doesn't hold;
6. similar to 4.; $A_k = \bigcup_{i=0}^{k} A_i$ (1.), so if $A_i$is infinite $\forall i \in \mathbb{N}$, then $A_k$is also infinite, so p_6 holds.