Computability Assignment
Year 2012/13 - Number 2

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1 Question

Let A, B be two sets. Prove that the properties below are equivalent.

- $A = \emptyset \vee B = \emptyset$
- $A \times B = \emptyset$

1.1 Answer

p: $(A = \emptyset \vee B = \emptyset) \iff (A \times B = \emptyset)$

($\Longrightarrow$) Suppose that $A = \emptyset$, the def of $A \times B = \{< x, y > | x \in A \wedge y \in B\} = C$ but by hypothesis A is empty so no such x exists. This implies that no couples can be formed to populate C, hence $C = A \times B = \emptyset$.

($\Longleftarrow$) If $A \times B = \emptyset$ then (taking the definition above) no couples $< x, y >$ exists. This implies that the formula $(x \in A \wedge y \in B)$ is false so one of the two statement is false $(x \notin A \vee y \notin B)$ but this is impossible by def of $A \times B$ so the only possibility is $A = \emptyset \vee B = \emptyset$.

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i \in \mathbb{N}}$, we define $\bigcup_{i=0}^{\infty} A_i = \bigcup\{A_i \mid i \in \mathbb{N}\}$ and $\bigcup_{i=0}^{k} A_i = \bigcup\{A_i \mid i \in \mathbb{N} \wedge i \leq k\} = A_0 \cup A_1 \cup \cdots \cup A_k$.

3 Question

Assume $(A_i)_{i \in \mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \cdots \subseteq \mathbb{N} \ (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcup_{i=0}^{k} A_i$
2. p_2: for all i, if A_i is infinite, then $A_i = A_{i+1}$
3. p_3: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$
4. p_4: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is finite
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite

3.1 Answer

Referring to the list above:

1. The hypothesis $(*)$ is sufficient to conclude that p_1 holds. By its definition: $A_k = A_0 \cup A_1 \cup \cdots \cup A_k$ so it contains all the numbers $i \leq k$.

2. The hypothesis $(*)$ is sufficient to conclude that p_2 does not hold. By its definition: $|A_i| < |A_{i+1}|$ and the element that makes the difference is exactly the $succ(i)$.

3. The hypothesis $(*)$ is sufficient to conclude that p_3 holds. By its definition $A_i = \bigcup\{A_i | i \in \mathbb{N}\}$ if we consider $i = \infty$ then $A_i = \mathbb{N}$.

4. The hypothesis $(*)$ is sufficient to conclude that p_4 does not hold. If we take $i = k$ then $\forall k \in \mathbb{N}.A_i = A_k$ this set is finite and contains k elements, but the $\bigcup_{i=0}^{\infty} A_i$ is infinite by def.

5. The hypothesis $(*)$ is sufficient to conclude that p_5 holds. Same explanation as in 4.

6. The hypothesis $(*)$ is sufficient to conclude that p_6 does not hold. Same explanation ad in 4.