Problem 1. Let A, B be two sets, P.T (i) $A = \emptyset \vee B = \emptyset$ (ii) $A \times B = \emptyset$ are equivalent

Proof. By definition $A \times B = \{(a, b) | a \in A, b \in B\}$

In order to prove (i) and (ii) are equivalent, we need to prove (i) $\Rightarrow$ (ii) and (ii) $\Leftarrow$ (i).

($\Leftarrow$) part: Suppose if $A = \emptyset$ or $B = \emptyset$, then by definition $A \times B = \emptyset$

($\Rightarrow$) part: Suppose if $A \times B = \emptyset$, and if by contradiction $A \neq \emptyset$ and $B \neq \emptyset$, then there exists (a, b) such that $a \in A$ and $b \in B$, hence $(a, b) \in A \times B$, and hence is a contradiction $\square$

Solution:

1. $p_1 : \forall k \in \mathbb{N}.A_k = \bigcup_{i=0}^{k} A_i$
   By definition for any two sets S_1, S_2, $S_1 = S_2$, if (a) $S_1 \subseteq S_2$ and (b) $S_2 \subseteq S_2$.
   (a) In our case by definition, of $\bigcup_{i=0}^{k} A_i$, $\forall k \in \mathbb{N}.A_k \subseteq \bigcup_{i=0}^{k} A_i$ is trivially true.
   (b) By contradiction, if $\exists k \in \mathbb{N}. \bigcup_{i=0}^{k} A_i \nsubseteq A_k$, then there exists $0 \leq j \leq k$ with $A_j \nsubseteq A_k$. But this is a contradiction to our hypothesis, and hence $\forall k \in \mathbb{N}. \bigcup_{i=0}^{k} A_i \subseteq A_k$. This implies that hypothesis is sufficient to deduce p_1 $\square$
2. p_2: forall i, if A_i is infinite, then $A_i = A_{i+1}$.
   Take the specific case when $A_0 = \mathbb{N} - 0$, and $A_i = \mathbb{N}$, for $i > 0$. Now for $i = 0$, although the hypothesis holds, and the premise of p_2, but not the conlusion of p_2. Hence p_2 does not hold.
   Take another specific case when $A_i = \mathbb{N}$, for $i \geq 0$. Now, p_2 holds in this case.
   Hence, the hypothesis is not sufficient to conclude anything about the truth of p_2
3. p_3: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$.
   Take the specific case when $A_i = \{j | j \in \mathbb{N}) \wedge 0 < j \leq i\}$, for all $i \in \mathbb{N}$, i.e. $A_0 = \emptyset, A_1 = \{1\}, A_2 = \{1, 2\}, \ldots.$ Now one can see that $0 \notin \bigcup_{i=0}^{\infty} A_i$, and p_3 does not hold.
   Take another specific case when $A_i = \{j | 0 \leq j \leq i\}$, $\forall i \in \mathbb{N}$, i.e. $A_0 = \{0\}, A_1 = \{0, 1\}, A_2 = \{0, 1, 2\}, \ldots.$ It is easy to see that $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$, and p_3 holds
   Hence, the hypothesis is not sufficient to conclude anything about the truth of p_3
4. p_4: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is finite.
   Take the specific case when $A_i = \{j | 0 \leq j \leq i\}$, $\forall i \in \mathbb{N}$, i.e. $A_0 = \{0\}, A_1 = \{0, 1\}, A_2 = \{0, 1, 2\}, \ldots.$ It is easy to see that $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$ and hence is infinite, where as, for any $i \in \mathbb{N}$, $|A_i| = i + 1$, and is finite.
   for a specific case where p_4 holds, see (5).
   Hence, the hypothesis is not sufficient to conclude anything about the truth of p_4

5. p_5: if $\forall i \in \mathbb{N}$. A_i is finite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite. Take the specific case when $A_i = \{0\}, \forall i \in \mathbb{N}$, i.e. $A_0 = \{0\}, A_1 = \{0\}, A_2 = \{0\}, ...$ It is easy to see that $\bigcup_{i=0}^{\infty} A_i = \{0\}$ and hence is finite.
   Hence, the hypothesis is not sufficient to conclude anything about the truth of p_5
6. p_6: if $\forall i \in \mathbb{N}$. A_i is infinite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite. For any $j \in \mathbb{N}$, since A_j is infinite, and since $A_j \subseteq \bigcup_{i=0}^{\infty} A_i$. Hence, $\bigcup_{i=0}^{\infty} A_i$ is infinite. Hence, the hypothesis is sufficient to conclude the truth of p_6