Computability Assignment
Year 2012/13 - Number 2

1 Question

Let A, B be two sets. Prove that the properties below are equivalent.

- $A = \emptyset \vee B = \emptyset$
- $A \times B = \emptyset$

1.1 Answer

Equivalence can be rewritten as $A = \emptyset \vee B = \emptyset \iff A \times B = \emptyset$. Lets us prove it:

$\Longrightarrow$) $A = \emptyset \vee B = \emptyset \Longrightarrow A \times B = \emptyset$:

Assuming that $A = \emptyset$, $\nexists x \in A. \forall y \in B. < x, y > \in A \times B \rightarrow A \times B = \emptyset$

$\Leftarrow$) $A \times B = \emptyset \Rightarrow A = \emptyset \vee B = \emptyset$:

$A \times B = \emptyset \Rightarrow (\nexists x \in A. \forall y \in B. < x, y > \in A \times B) \vee (\nexists y \in B. \forall x \in A. < x, y > \in A \times B) \Rightarrow A = \emptyset \vee B = \emptyset$

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i \in \mathbb{N}}$, we define $\bigcup_{i=0}^{\infty} A_i = \bigcup \{A_i \mid i \in \mathbb{N}\}$ and $\bigcup_{i=0}^{k} A_i = \bigcup \{A_i \mid i \in \mathbb{N} \ \wedge \ i \leq k\} = A_0 \cup A_1 \cup \cdots \cup A_k$.

3 Question

Assume $(A_i)_{i \in \mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \cdots \subseteq \mathbb{N} \ (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or

- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcup_{i=0}^{k} A_i$
2. p_2: for all i, if A_i is infinite, then $A_i = A_{i+1}$
3. p_3: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$
4. p_4: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is finite
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite

3.1 Answer

Supposing $\mathrm{H} = A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \cdots \subseteq \mathbb{N}$

- p1) p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcup_{i=0}^{k} A_i$. Since, for the hypothesis, each $A_i \subseteq A_{i+1} \wedge A_i \cup A_{i+1} = A_{i+1} \Rightarrow (\bigcup_{i=0}^{k-1} A_i) \cup A_k = A_k$. The property holds under H.
- p2)p_2: for all i, if A_i is infinite, then $A_i = A_{i+1}$. Under H : $A_i \subseteq A_{i+1} \Rightarrow |A_i| \leq |A_{i+1}|$ hence if $A_i$is infinite, $A_{i+1}$is infinite too. The property holds.
- p3)p_3: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$. Taking into account H, we can assume that $A_i \neq A_{i+1} \Longleftrightarrow A_i \supseteq A_{i+1} \wedge A_i \subseteq A_{i+1}$. By absurd, we make the hypothesis that: $A_i \neq A_{i+1} \Rightarrow A_i \supsetneq A_{i+1} \wedge A_i \nsubseteq A_{i+1} \Rightarrow A_i \nsubseteq A_{i+1}$ which is false under H, then due to the contraddiction the property does not hold in H.
- p4)p_4: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is finite: Taking into account H, we can infer that since A_i is finite and $A_{i-1} \subseteq A_i$, then A_{i-1} is finite too. Considering $A_i \cup A_{i+1} = A_{i+1}$ (from H): $\forall i \in \mathbb{N}.|\bigcup_{i=0}^{k} A_i| = |A_k| \Rightarrow |\bigcup_{i=0}^{k} A_i| = |A_\infty| = |\mathbb{N}|$which is infinite. This property does not hold under H.
- p5)p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite: $\forall i \in \mathbb{N}.\ A_i$ is finite and $A_{i-1} \subseteq A_i \Rightarrow A_{i-1}$ is finite. From $|A_{i-1} \cup A_i| \leq |A_i|$ we can state that$|\bigcup_{i=0}^{\infty} A_i| \leq |A_\infty| = |\mathbb{N}|$ which is infinite. The property holds in H.
- p6)p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite: stated that, from H : $A_{i-1} \subseteq A_i \Rightarrow |A_{i-1} \cup A_i| \leq |A_i|$ we can conclude that $|\bigcup_{i=0}^{\infty} A_i| = |A_\infty|$ where $|A_\infty|$ is infinite. The property holds.