Computability Assignment
Year 2012/13 - Number 2

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1 Question

Let A, B be two sets. Prove that the properties below are equivalent.

- $A = \emptyset \vee B = \emptyset$
- $A \times B = \emptyset$

1.1 Answer

Let's prove that $A \times B = \emptyset \Rightarrow A = \emptyset \vee B = \emptyset$.

Let's take $A \times B = \emptyset$, where $A \times B = \{\langle a, b\rangle \,|a \epsilon A \wedge b \epsilon B\}$. We have three cases:

a) if $A \neq \emptyset \wedge B \neq \emptyset$, then the cartesian product contains all the pairs of the combinations between elements of A and elements of B. No set is empty, so $A \times B \neq \emptyset$

b) if $A = \emptyset \wedge B = \emptyset$, then $A \times B = \emptyset$, because no set has elements to combine

c) if $A = \emptyset \wedge B \neq \emptyset$OR $A \neq \emptyset \wedge B = \emptyset$, then $A \times B = \emptyset$, because the combination of an empty set with a no-empty set is empty

So $A \times B = \emptyset$in the cases: $A = \emptyset \wedge B = \emptyset$, $A = \emptyset \wedge B \neq \emptyset$, $A \neq \emptyset \wedge B = \emptyset$, which can be expressed as: $A = \emptyset \vee B = \emptyset$

Now let's prove that $A \times B = \emptyset \Leftarrow A = \emptyset \vee B = \emptyset$.

If $A = \emptyset \vee B = \emptyset$, we are in the cases b) and c) described before, and for all of them $A \times B = \emptyset$, as already proved.

So, $A \times B = \emptyset \Leftrightarrow A = \emptyset \vee B = \emptyset$.

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i\in\mathbb{N}}$, we define $\bigcup_{i=0}^{\infty} A_i = \bigcup\{A_i \mid i \in \mathbb{N}\}$ and $\bigcup_{i=0}^{k} A_i = \bigcup\{A_i \mid i \in \mathbb{N} \ \wedge \ i \leq k\} = A_0 \cup A_1 \cup \cdots \cup A_k$.

3 Question

Assume $(A_i)_{i\in\mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \cdots \subseteq \mathbb{N} \ (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcup_{i=0}^{k} A_i$
2. p_2: for all i, if A_i is infinite, then $A_i = A_{i+1}$
3. p_3: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$
4. p_4: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is finite
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite

3.1 Answer

1. $\forall k \in \mathbb{N}.\ A_k = \bigcup_{i=0}^{k} A_i$
$\bigcup_{i=0}^{k} A_i = A_0 \bigcup A_1 \bigcup ... \bigcup A_k$
if $A_0 \subseteq A_1 \subseteq ... \subseteq A_k$, then $A_k$already includes the unione of the A_is, and the union of them is equal to A_k
TRUE: the hypothesis is sufficient to conclude that p_1 holds

2. for all i, if A_i is infinite, then $A_i = A_{i+1}$
this is not true. Here there is a counterexample:
let's take $A_i = \mathbb{N}\backslash\{0\}$, and $A_{i+1} = \mathbb{N}$ (which is valid: $A_i \subseteq A_{i+1}$)
both the sets are infinite, but $A_i \neq A_{i+1}$
FALSE: the hypothesis is sufficient to conclude that $p_2$does not hold

3. if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$

hypothesis: if $A_i \neq A_{i+1}$, then we have strict inclusions: $A_1 \subset A_2 \subset ...$

the last $A_i$is included in $\mathbb{N}$, but we cannot know if the union of the A_is gives $\mathbb{N}$

for instance, if $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 1, 2\}$and so on, it is true; but if $A_0 = \{1\}$, $A_1 = \{1, 3\}$, $A_2 = \{1, 3, 5\}$ and so on, it is false

UNKNOWN: the hypothesis is not sufficient to conclude anything

4. if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is finite

if every $A_i$is finite, then the union $A_0 \bigcup A_1 \bigcup ...$ is an infinite operation, but its result will always be finite

for instance, the union of the finite sets $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 1, 2\}, \ldots$ can only have a finite number of elements

TRUE: the hypothesis is sufficient to conclude that $p_4$holds

5. if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite

it is the opposite of propery 4, so, false: the infinite union of finite sets is finite and not infinite

it is not possible to create the infinite from finite elements

FALSE: the hypothesis is sufficient to conclude that $p_5$holds

6. if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite

this is true: it two or more sets are infinite, their union is infinite as well

for instance, if $A_0 = \{100, 101, ...\}$, $A_1 = \{99, 100, 101, ...\}$, $A_2 = \{98, 99, 100, 101, ...\}, \ldots$, the result of the union is infinite

TRUE: the hypothesis is sufficient to conclude that $p_6$holds