Computability Assignment
Year 2012/13 - Number 2

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1 Question

Let A, B be two sets. Prove that the properties below are equivalent.

- $A = \emptyset \vee B = \emptyset$
- $A \times B = \emptyset$

Answer

$A = \emptyset \iff \neg\exists x.x \in A \iff \forall x.x \notin A$
$B = \emptyset \iff \neg\exists y.y \in B \iff \forall y.y \notin B$
$A \times B = \emptyset \iff \{(x, y) \mid x \in A, y \in B\} = \emptyset \iff$
$\neg\exists\,(x, y)\,.x \in A \wedge y \in B \iff$
$\forall\,(x, y)\,.x \notin A \vee y \notin B \iff$
$\forall x \forall y.x \notin A \vee y \notin B \iff$
$\forall x.x \notin A \vee \forall y.y \notin B \iff$
$A = \emptyset \vee B = \emptyset$

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i\in\mathbb{N}}$, we define $\bigcup_{i=0}^{\infty} A_i = \bigcup\{A_i \mid i \in \mathbb{N}\}$ and $\bigcup_{i=0}^{k} A_i = \bigcup\{A_i \mid i \in \mathbb{N} \ \wedge\ i \leq k\} = A_0 \cup A_1 \cup \cdots \cup A_k$.

3 Question

Assume $(A_i)_{i\in\mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \cdots \subseteq \mathbb{N}\ (*)$$

For each property p_i shown below, state whether

- X: the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- Y: the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- Z: the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcup_{i=0}^{k} A_i$
2. p_2: for all i, if A_i is infinite, then $A_i = A_{i+1}$
3. p_3: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$
4. p_4: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is finite
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite

3.1 Answer

1. X: $A_k = \bigcup_{i=0}^{k} A_i \Longleftrightarrow A_k \subseteq \bigcup_{i=0}^{k} A_i$(trivial) and $\bigcup_{i=0}^{k} A_i \subseteq A_k$. Hypothesis $A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \cdots \subseteq \mathbb{N}$ is sufficient to conclude that $\bigcup_{i=0}^{k} A_i \subseteq A_k$ and respectively that p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcup_{i=0}^{k} A_i$ holds.
2. todo
3. Z: We can't conclude X and we can't conclude Y, therefore Z.
   (a) X: counterexample: $A_k = \{2 * i | i \leq k\}$ shows that (*) holds and $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, but p_3 does not hold, that is (*) is not sufficient to conclude that $p_3$holds.
   (b) Y: $p_3$does not hold $\Longleftrightarrow \neg p_3$ holds. (*) is sufficient to conclude that $\neg p_3$ holds. $\neg p_3 = \forall i \in \mathbb{N}.\ A_i \neq A_{i+1} \wedge \bigcup_{i=0}^{\infty} A_i \neq \mathbb{N}$. Counterexample: $A_k = \{i | i \leq k\}$ shows that (*) holds and $\neg p_3$does not hold, therefore p_3 holds.
4. Z: We can't conclude X and we can't conclude Y, therefore Z.
   (a) X: counterexample: $A_k = \{i | i \leq k\}$ shows that (*) holds and $\forall i \in \mathbb{N}.\ | A_i | < \infty$; but $| \bigcup_{i=0}^{\infty} A_i | = \infty$. p_4 does not hold, that is (*) is not sufficient to conclude that $p_4$holds.
   (b) Y: counterexample: $A_k = \{0\}$ shows that (*) holds and $\forall i \in \mathbb{N}.\ | A_i | < \infty$; but$| \bigcup_{i=0}^{\infty} A_i | \neq \infty$. p_4 holds, that is (*) is not sufficient to conclude that $p_4$does not hold.
5. Z: We can't conclude X and we can't conclude Y, therefore Z.

   (a) see (4.b).
   (b) see (4.a).

6. Holds regardless of (*).