Computability Assignment
Year 2012/13 - Number 2

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1 Question

Let A, B be two sets. Prove that the properties below are equivalent.

- $A = \emptyset \vee B = \emptyset$
- $A \times B = \emptyset$

1.1 Answer

Definition of cartesian product: $A \times B = \{\langle a, b\rangle | a \in A \wedge b \in B\}$

- Proof of $A = \emptyset \vee B = \emptyset \Longrightarrow A \times B = \emptyset$: if one of the two initial sets is empty, in this case A, $A = \emptyset \Longrightarrow \neg\exists a \in A \Longrightarrow$(by definition of cartesian product)$\neg\exists\langle a, b\rangle \in A \times B \Longrightarrow A \times B = \emptyset$. Similarly, this also applies to B.
- Proof of $A \times B = \emptyset \Longrightarrow A = \emptyset \vee B = \emptyset$: by definition of cartesian product, $\neg\exists\langle a, b\rangle . a \in A \wedge b \in B \Longrightarrow \neg(\exists a \in A \wedge \exists b \in B) \Longrightarrow \neg\exists a \in A \vee \neg\exists b \in B \Longrightarrow A = \emptyset \vee B = \emptyset$.

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i\in\mathbb{N}}$, we define $\bigcup_{i=0}^{\infty} A_i = \bigcup\{A_i \mid i \in \mathbb{N}\}$ and $\bigcup_{i=0}^{k} A_i = \bigcup\{A_i \mid i \in \mathbb{N} \ \wedge \ i \leq k\} = A_0 \cup A_1 \cup \cdots \cup A_k$.

3 Question

Assume $(A_i)_{i\in\mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \cdots \subseteq \mathbb{N} \ (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcup_{i=0}^{k} A_i$
2. p_2: for all i, if A_i is infinite, then $A_i = A_{i+1}$
3. p_3: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$
4. p_4: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is finite
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite

3.1 Answer

1. True, because $a \in A_k \iff a \in \bigcup_{i=0}^{k} A_i$, in fact:
   (a) Left direction: the union contains $A_k$by definition.
   (b) Right direction: all the elements of A_k are contained in the union, and no other elements are there. Infact, $\forall b \in \mathbb{N} \mid b < n.\ A_k \supseteq A_b$.
2. False. For example, $A_1 = \{2a,\ a \in \mathbb{N}|\}$, $A_2 = \{a,\ a \in \mathbb{N}\}$. The first set is infinite, the second contains all the elements of the first one, but the second one also contains elements that the first one does not contain (odd integers).
3. False. For example $A_i = \{a \in N \mid 1 \leq a \leq i+1\}$. The union will be equal to $\mathbb{N}^*$.
4. False. For example, the sets defined in the previous point are all finite, but the union is infinite.
5. False. For example, the sets could be all equal to each other and finite. The union would still be the same set.
6. True, because $A_0 \in \bigcup_{i=0}^{\infty} A_i$ and $A_0$is infinite