

# Computability Assignment

## Year 2012/13 - Number 2

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### 1 Question

Let  $A, B$  be two sets. Prove that the properties below are equivalent.

- $A = \emptyset \vee B = \emptyset$
- $A \times B = \emptyset$

#### 1.1 Answer

$$|A \times B| = |A||B| \implies (A \times B = \emptyset \iff |A \times B| = 0) \iff |A| = 0 \vee |B| = 0 \implies A = \emptyset \vee B = \emptyset$$

### 2 Preliminaries

Given an infinite sequence of sets  $(A_i)_{i \in \mathbb{N}}$ , we define  $\bigcup_{i=0}^{\infty} A_i = \bigcup \{A_i \mid i \in \mathbb{N}\}$  and  $\bigcup_{i=0}^k A_i = \bigcup \{A_i \mid i \in \mathbb{N} \wedge i \leq k\} = A_0 \cup A_1 \cup \dots \cup A_k$ .

### 3 Question

Assume  $(A_i)_{i \in \mathbb{N}}$  to be an infinite sequence of sets of natural numbers, satisfying

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \cdots \subseteq \mathbb{N} \quad (*)$$

For each property  $p_i$  shown below, state whether

- the hypothesis  $(*)$  is sufficient to conclude that  $p_i$  holds; or
- the hypothesis  $(*)$  is sufficient to conclude that  $p_i$  does not hold; or

- the hypothesis  $(*)$  is not sufficient to conclude anything about the truth of  $p_i$ .

Justify your answers (briefly).

1.  $p_1$ :  $\forall k \in \mathbb{N}. A_k = \bigcup_{i=0}^k A_i$
2.  $p_2$ : for all  $i$ , if  $A_i$  is infinite, then  $A_i = A_{i+1}$
3.  $p_3$ : if  $\forall i \in \mathbb{N}. A_i \neq A_{i+1}$ , then  $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$
4.  $p_4$ : if  $\forall i \in \mathbb{N}. A_i$  is finite, then  $\bigcup_{i=0}^{\infty} A_i$  is finite
5.  $p_5$ : if  $\forall i \in \mathbb{N}. A_i$  is finite, then  $\bigcup_{i=0}^{\infty} A_i$  is infinite
6.  $p_6$ : if  $\forall i \in \mathbb{N}. A_i$  is infinite, then  $\bigcup_{i=0}^{\infty} A_i$  is infinite

### 3.1 Answer

1. true (thanks to the given definition)
2. hypothesis not sufficient
3. false. Counter-example:  $A_n = \{n+1\} \implies \bigcup_{i=0}^{\infty} A_i = \mathbb{N} \setminus \{0\}$
4. true because  $A_1 \subseteq \dots \subseteq A_n \implies A_1 \cup \dots \cup A_n = A_n$  and  $A_n$  is finite
5. false because contradiction of .4
6. true (same motivation of .4)