Computability Assignment
Year 2012/13 - Number 2

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1 Question

Let A, B be two sets. Prove that the properties below are equivalent.

- $A = \emptyset \vee B = \emptyset$
- $A \times B = \emptyset$

1.1 Answer

for $A = \emptyset \vee B = \emptyset$
Sure $A = B = \emptyset$. (RZ: no)
and $A = B = \emptyset \iff A = \emptyset \vee B = \emptyset$

for $A \times B = \emptyset \iff A = B = \emptyset$ **(?)** So the two properties are equivalent

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i \in \mathbb{N}}$, we define $\bigcup_{i=0}^{\infty} A_i = \bigcup\{A_i \mid i \in \mathbb{N}\}$ and $\bigcup_{i=0}^{k} A_i = \bigcup\{A_i \mid i \in \mathbb{N} \wedge i \leq k\} = A_0 \cup A_1 \cup \cdots \cup A_k$.

3 Question

Assume $(A_i)_{i \in \mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \cdots \subseteq \mathbb{N} \ (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcup_{i=0}^{k} A_i$
2. p_2: for all i, if A_i is infinite, then $A_i = A_{i+1}$
3. p_3: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$
4. p_4: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is finite
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite

3.1 Answer

1. True
   $\bigcup_{i=0}^{k} A_i = \bigcup\{A_i \mid i \in \mathbb{N} \ \wedge \ i \leq k\} = A_0 \cup A_1 \cup \cdots \cup A_k$
   and $A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \cdots \subseteq \mathbb{N}$ $(*)$
   $A_0 \cup A_1 \cup \cdots \cup A_k = A_k$
   so $\forall k \in \mathbb{N}.\ A_k = \bigcup_{i=0}^{k} A_i$

2.Not sufficient to conclude anything about the truth Assume $A_i = A_{i+1} = \mathbb{N}$,then it is right

however,if $A_0$is a set of all odds,$A_1 = A_0 \cup \{2\}$,then it is false

3.Not sufficient to conclude anything about the truth Assume $A_0$is a set of all odds,$A_1 = A_0 \cup \{0\}, A_2 = A_0 \cup \{0, 2\}, A_3 = A_0 \cup \{0, 2, 4\}$......$A_\infty = A_0 \cup \{2n\} = \mathbb{N}$.So,it is right.

however, if $A_0$is a set of all odds expect 1,the other $A_i$is defined as previous,then $A_\infty = \mathbb{N} \setminus \{1\} \neq \mathbb{N}$

4.Not sufficient to conclude anything about the truth if $\forall i \in \mathbb{N}.\ A_i = \{1\}$,$\bigcup_{i=0}^{\infty} A_i = \{1\}$,is finite.In this case,it is true.

hoever,if $A_i = \{x | x \leqslant i, i \in \mathbb{N}\}$,$\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$ is infinite,it is wrong!

5.Not sufficient to conclude anything about the truth proof: the same with 4.

6.True for $A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \cdots \subseteq \mathbb{N}$ $(*)$

$A_0 \subseteq \bigcup_{i=0}^{\infty} A_i$

$A_0$is infinite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite.