Computability Assignment
Year 2012/13 - Number 2

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1 Question

Let A, B be two sets. Prove that the properties below are equivalent.

- $A = \emptyset \vee B = \emptyset$ (p)
- $A \times B = \emptyset$ (q)

1.1 Answer

1. $(p \to q)$ By contradiction, if $A \times B \neq \emptyset$, then $\exists(a, b) \in A \times B$, that is, $a \in A \wedge b \in B$, which is false.
2. $(p \leftarrow q)$ Also by contradiction. i.e. $A \neq \emptyset \wedge B \neq \emptyset$, then $\exists a \in A \wedge \exists b \in B$, which makes $(a, b) \in A \times B$, again false.

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i \in \mathbb{N}}$, we define $\bigcup_{i=0}^{\infty} A_i = \bigcup\{A_i \mid i \in \mathbb{N}\}$ and $\bigcup_{i=0}^{k} A_i = \bigcup\{A_i \mid i \in \mathbb{N} \ \wedge \ i \leq k\} = A_0 \cup A_1 \cup \cdots \cup A_k$.

3 Question

Assume $(A_i)_{i \in \mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \cdots \subseteq \mathbb{N} \ (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or

- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcup_{i=0}^{k} A_i$
2. p_2: for all i, if A_i is infinite, then $A_i = A_{i+1}$
3. p_3: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$
4. p_4: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is finite
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite

3.1 Answer

1. True. By induction. (base) $\bigcup_{i=1}^{1} A_i = A_1$, (inductive) $A_{n-1} \subseteq A_n \to \bigcup_{i=1}^{n} A_i = \bigcup_{i=i}^{n-1} A_i \cup A_n = A_{n-1} \cup A_n = A_n$
2. Indecisive. We may have $A_1 = \{1\} = A_2 = \cdots = A_n \subsetneq A_{n+1} = \{1, 2\} \subseteq \cdots \subseteq \mathbb{N}$
3. Indecisive. Choose $A_i = \{2k | 0 < k \leqslant i,\ k \in \mathbb{N}\}$, we will get the set of all even numbers.
4. True. According to 1. (RZ: no)
5. False. (RZ: might also be true)
6. True.