Computability Assignment
Year 2012/13 - Number 2

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1 Question

Let A, B be two sets. Prove that the properties below are equivalent.

- $A = \emptyset \vee B = \emptyset$
- $A \times B = \emptyset$

1.1 Answer

Let's define the three sets above :

- $A = \{a | a \in A\}$
- $B = \{b | b \in B\}$
- $A \times B = \{< a, b > | a \in A \wedge b \in B\}$

Using the definition of $A \times B$ above the cartesian product of two set is not empty if and only if the $\exists a \in A \wedge \exists b \in B$ a set can be only empty or not empty so an empty set can be seen as a"not not empty set" :

- $\neg(\exists a \in A \wedge \exists b \in B)$

by apply simples logical rules we obtain that this is equal to $\neg\exists a \in A \vee \neg\exists b \in B$ that in terms of sets (defined as above means that) :

- $A = \emptyset \vee B = \emptyset$

so we have proved that $A \times B = \emptyset \Longrightarrow A = \emptyset \vee B = \emptyset$

the inverse implication can be seen in the same way applaing a negation to$\neg\exists a \in A \vee \neg\exists b \in B$

obtaining $\exists a \in A \wedge \exists b \in B$ that is the condition in wich a cartesian product set exists.

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i\in\mathbb{N}}$, we define $\bigcup_{i=0}^{\infty} A_i = \bigcup\{A_i \mid i \in \mathbb{N}\}$ and $\bigcup_{i=0}^{k} A_i = \bigcup\{A_i \mid i \in \mathbb{N} \wedge i \leq k\} = A_0 \cup A_1 \cup \cdots \cup A_k$.

3 Question

Assume $(A_i)_{i\in\mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \cdots \subseteq \mathbb{N} \ (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcup_{i=0}^{k} A_i$
2. p_2: for all i, if A_i is infinite, then $A_i = A_{i+1}$
3. p_3: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$
4. p_4: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is finite
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite

3.1 Answer

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcup_{i=0}^{k} A_i$

   start defining some basic set rules

   if $A \subseteq B$

   $A \cup B = B$

   so by the definition of above we have that for istance ...$A_{k-2} \subseteq A_{k-1} \subseteq A_k$ that means that all the sets twith $i < k$ are contained in A_k so the union of all this sets are A_k so p1 is holds;

2. p_2: for all i, if A_i is infinite, then $A_i = A_{i+1}$

   for the definition if A_{i+1} have to contain A_i or can be equals to A_{i+1} so we can say that if A_i is infinite A_{i+1} has to be infinite so by the definition above we have that $A_i = A_{i+1}$ so the first condition to make two set equals is that $|A_i| = |A_{i+1}|$ that is not true for ol the cases in fact if i took A_i as $\mathbb{N} \setminus \{0\}$ is an infinite set and if we define A_{i+1} as whole $\mathbb{N}$ we see that this two set can exists by our preliminarities but are not equal. Exists one other case that if $A_0 = \mathbb{N}$ each consecutive set are equal to the previous one so $A_\infty = \mathbb{N}$ in fact the super set A_{i+1} cannot be greater than A_i because is equal to $A_i = \mathbb{N}$ cannot exists a set of natural number greater than $\mathbb{N}$

3. p_3: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$

   this statement tell us that if $A_i \neq A_{i+1}$ at each step $A_i \subset A_{i+1}$ intuitlively you can say this sets can cover whole the natural number set but it's not strictly true because if at each step we add for istance an even number when we reach the A_{inf} we have an infinite set that is $\mathbb{N} \backslash \{odds\}$ and not the whole $\mathbb{N}$. so we cant say anything about this preposition.

4. p_4: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is finite

   if all the sets are finite the union of all this set up to infinite are finite and is the biggest (RZ: there is no "larger" object in an infinite sequence, in general) of all this finite set by the definition of A_i seen in the preliminaries. So this is true.

5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite

   this can be valid only if $p4$ is false because a set cannot be either finite and infinite so this is false.

6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite

   if all the set A_i are infinite their union are the greater of this set but it still infinite so this definition is true.