Computability Assignment
Year 2012/13 - Number 2

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1 Question

Let A, B be two sets. Prove that the properties below are equivalent.

- $A = \emptyset \vee B = \emptyset$

- $A \times B = \emptyset$

Answer

$A = \emptyset \Longleftrightarrow \neg \exists x.x \in A \Longleftrightarrow \forall x.x \notin A$

$B = \emptyset \Longleftrightarrow \neg \exists y.y \in B \Longleftrightarrow \forall y.y \notin B$

$A \times B = \emptyset \Longleftrightarrow \{(x,y) \mid x \in A, y \in B\} = \emptyset \Longleftrightarrow$

$\neg \exists (x,y).x \in A \wedge y \in B \Longleftrightarrow$

$\forall (x,y).x \notin A \vee y \notin B \Longleftrightarrow$

$\forall x \forall y.x \notin A \vee y \notin B \Longleftrightarrow$

$\forall x.x \notin A \vee \forall y.y \notin B \Longleftrightarrow$

$A = \emptyset \vee B = \emptyset$

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i \in \mathbb{N}}$, we define $\bigcup_{i=0}^{\infty} A_i = \bigcup\{A_i \mid i \in \mathbb{N}\}$
and $\bigcup_{i=0}^{k} A_i = \bigcup\{A_i \mid i \in \mathbb{N} \wedge i \leq k\} = A_0 \cup A_1 \cup \cdots \cup A_k$.

3 Question

Assume $(A_i)_{i \in \mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \cdots \subseteq \mathbb{N} \; (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or

- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or

- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcup_{i=0}^{k} A_i$

2. p_2: for all i, if A_i is infinite, then $A_i = A_{i+1}$

3. p_3: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$

4. p_4: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is finite

5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite

6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite

3.1 Answer

TODO