

Computability Assignment

Year 2012/13 - Number 1

Please keep this file anonymous: do not write your name inside this file.

More information about assignments at <http://disi.unitn.it/~zunino/teaching/computability/assignments>

1 Question

Define a binary property $p(x, y)$ over natural numbers such that we have both

1. $\forall x \in \mathbb{N}. \exists y \in \mathbb{N}. p(x, y) \iff \neg \exists x \in \mathbb{N}. \forall y \in \mathbb{N}. \neg p(x, y)$
2. $\neg \exists y \in \mathbb{N}. \forall x \in \mathbb{N}. p(x, y) \iff \forall y \in \mathbb{N}. \exists x \in \mathbb{N}. \neg p(x, y)$

Provide a definition for p , and a proof for the above claims.

NOTE: this is the *third* review of the document, since two major errors have been found.
I apologize for the multiple versions loaded.

LOG CHANGES:

1. changed “the second statement is obviously wrong” into “the second statement is obviously wrong”
2. changed “anti-symmetric” ($\forall a, b \in A. aRb \wedge bRa \implies a = b$) into “a-symmetric” ($\forall a, b \in A. aRb \implies \neg(bRa)$), since the latter was the intended meaning and the first doesn’t apply to the problem.

1.1 First Trial:

Let $\langle x, y \rangle \in p \iff y = f(x)$, where f is a recursive function so defined:

$$f(x) = \begin{cases} y = x/2 & \text{if } \exists! n \in \mathbb{N}. x = 2 * n \\ y = (x - 1)/2 & \text{if } \exists! n \in \mathbb{N}. x = 2 * n + 1 \end{cases}, \text{ where it is implicit that}$$

$y \in \mathbb{N}$ and that the symbols $+$, $-$, $/$ follow the usual arithmetic semantics among natural numbers.

STATEMENT 1: (PROOF BY ENUMERATION OF CASES)

Let $x_0 \in \mathbb{N}$ be an arbitrarily chosen number, there are two possible cases:
 x_0 may be either even or odd.

CASE A: x_0 is even.

If x_0 is even, there exists by definition a (unique) $n \in \mathbb{N}$ such that $x_0 = 2 * n$.
Therefore by the actual definition of function f , we have that $y_0 = f(x_0) = x_0/2$, and being more precise $y_0 = f(x_0) = f(2 * n) = 2 * n/2 = n$.

Since $y_0 = n \in \mathbb{N}$, the tuple $\langle x_0, y_0 \rangle$ belongs to the binary relationship p .

CASE B: x_0 is odd.

If x_0 is odd, there exists by definition a (unique) $n \in \mathbb{N}$ such that $x_0 = 2 * n + 1$.

Therefore by the actual definition of function f , we have that $y_0 = f(x_0) = (x_0 - 1)/2$ which means $y_0 = f(x_0) = f(2 * n + 1) = ((2 * n + 1) - 1)/2 = (2 * n)/2 = n$.

Since $y_0 = n \in \mathbb{N}$, the tuple $\langle x_0, y_0 \rangle$ belongs to the binary relationship p .

CONCLUSION:

Having chosen $x_0 \in \mathbb{N}$ arbitrarily and examined all the possible cases, we may generalize and say that for each $x \in \mathbb{N}$ there exists $y \in \mathbb{N}$ such that $p(x, y)$, which is precisely what Statement 1 is standing for.

STATEMENT 2:

Let $y \in \mathbb{N}$ be, for absurdum, a chosen number such that $\forall x \in \mathbb{N}. p(x, y)$.

Since the property holds for all $x \in \mathbb{N}$, it certainly has to hold for a strict subset of \mathbb{N} , let's say $A_m = \{x | x = m \vee x = s(m) \vee x = s(s(m))\}$ ($s(x)$ is the usual successor function).

We may take an arbitrary set $A_m = \{m, m + 1, m + 2\}$, hence we make no restriction and use it directly without instantiating m .

For hypotheses $\langle m, y \rangle \in p$, $\langle m + 1, y \rangle \in p$ and $\langle m + 2, y \rangle \in p$.

Our goal is to show that the following statement $y = f(m) = f(m + 1) = f(m + 2)$ is false, thus falsifying the entire conjecture.

There are two cases:

$$\text{if } m = 2 * k \text{ then } \begin{cases} f(m) = f(2 * k) = 2 * k/2 & = k \\ f(m + 1) = f(2 * k + 1) = (2 * k + 1 - 1)/2 & = k \\ f(m + 2) = f(2 * k + 2) = (2 * k + 2)/2 & = k + 1 \end{cases},$$

which clearly contradicts the hypothesis.

$$\text{if } m = 2 * k + 1 \text{ then } \begin{cases} f(m) = f(2 * k + 1) = (2 * k + 1 - 1)/2 & = k \\ f(m + 1) = f(2 * k + 2) = (2 * k + 2)/2 & = k + 1, \\ f(m + 2) = f(2 * k + 3) = (2 * k + 3 - 1)/2 & = k + 1 \end{cases},$$

which clearly contradicts the hypothesis.

Therefore there exists no $y \in \mathbb{N}$ with such properties, therefore we can discharge the initial hypothesis and conclude $\neg \exists y \in \mathbb{N}. \forall x \in \mathbb{N}. p(x, y)$.

1.2 Second Trial.

Let $\langle x, y \rangle \in p \iff y = s(x)$, so that p may be read as the property relationship of “ x is the predecessor of y ” or, conversely, “ y is the successor of x ”. The function $s(x)$ is total and injective, while the property p is anti-reflexive, a-symmetric and clearly not transitive [proof by exercise].

1:

The first statement is straightforward, since \mathbb{N} is a well-founded set that may be recursively defined as

$0 \in \mathbb{N}$ and $\forall x \in \mathbb{N} \Rightarrow s(x) \in \mathbb{N}$

then it is obvious that $\forall x \in \mathbb{N}$, its successor $s(x)$ belongs to the set of \mathbb{N} .

2:

The second statement is obviously right, since otherwise it would violate both the injective property of $s(x)$ and the anti-reflexive and a-symmetric properties of p .

Let's try to sort out a proof:

Let $y \in \mathbb{N}$ be, for absurdum, a chosen number such that $\forall x \in \mathbb{N}. p(x, y)$.

Since $y \in \mathbb{N}$, by the property of \mathbb{N} there exist a (possibly large) finite index $n \in \mathbb{N}$ such that $y = s_n(s_{n-1}(\dots s_1(0)\dots))$. Also, since $y \in \mathbb{N}$, there must exist $y' \in \mathbb{N}$ such that $y' = s(y)$, which means $\langle y, y' \rangle \in p$.

For hypothesis y is such that $\forall x \in \mathbb{N}. p(x, y)$, therefore also $\langle y', y \rangle \in p$ and $y = s(y')$.

Hence one could deduct $y = s(y') = s(s(y))$, which follows into:

$$y = s_n(s_{n-1}(\dots s_1(0)\dots)) = s_{n+2}(s_{n+1}(\dots s_1(0)\dots))) = s(s(y))$$

This is clearly possible only if $n = n + 2$, but clearly $0 \neq 2$.

Since the only hypothesis made was the existence of a $y \in \mathbb{N}$ with such satisfying the second statement, and it was shown that the property didn't hold for an arbitrarily chosen y , we can discard the claim and deduce $\neg \exists y \in \mathbb{N}. \forall x \in \mathbb{N}. p(x, y)$ is true.

Viger