Computability Assignment
Year 2012/13 - Number 1

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1 Question

Define a binary property $p(x, y)$ over natural numbers such that we have both

1. $\forall x \in \mathbb{N}.\exists y \in \mathbb{N}.p(x, y) \iff \neg\exists x \in \mathbb{N}.\forall y \in \mathbb{N}.\neg p(x, y)$
2. $\neg\exists y \in \mathbb{N}.\forall x \in \mathbb{N}.p(x, y) \iff \forall y \in \mathbb{N}.\exists x \in \mathbb{N}.\neg p(x, y)$

Provide a definition for p, and a proof for the above claims.

1.1 First Trial:

Let $< x, y >\in p \iff y = f(x)$, where f is a recursive function so defined:
$f(x) = \begin{cases} y = x/2 & if\ \exists! n \in \mathbb{N}.x = 2 * n \\ y = (x-1)/2 & if\ \exists! n \in \mathbb{N}.x = 2 * n + 1 \end{cases}$, where it is implicit that $y \in \mathbb{N}$ and that the symbols $+, -, /$follow the usual arithmetic semantics among natural numbers.

Statement 1: (proof by enumeration of cases)

Let $x_0 \in \mathbb{N}$ be an arbitrarily chosen number, there are two possible cases: x_0 may be either even or odd.

Case A: $x_0$is even.

If x_0 is even, there exists by definition a (unique) $n \in \mathbb{N}$ such that $x_0 = 2*n$.

Therefore by the actual definition of function f, we have that $y_0 = f(x_0) = x_0/2$, and being more precise $y_0 = f(x_0) = f(2*n) = 2*n/2 = n$.

Since $y_0 = n \in \mathbb{N}$, the tuple $< x_0, y_0 >$belongs to the binary relationship p.

Case B: x_0 is odd.

If x_0 is odd, there exists by definition a (unique) $n \in \mathbb{N}$such that $x_0 = 2*n+1$.

Therefore by the actual definition of function f, we have that $y_0 = f(x_0) = (x_0-1)/2$ which means $y_0 = f(x_0) = f(2*n+1) = ((2*n+1)-1)/2 = (2*n)/2 = n$.

Since $y_0 = n \in \mathbb{N}$, the tuple $< x_0, y_0 >$belongs to the binary relationship p.

CONCLUSION:

Having chosen $x_0 \in \mathbb{N}$ arbitrarily and examinated all the possible cases, we may generalize and say that for each $x \in \mathbb{N}$ there exists $y \in \mathbb{N}$ such that $p(x, y)$, which is precisely what Statement 1 is standing for.

STATEMENT 2:

Let $y \in \mathbb{N}$ be, for absurdum, a chosen number such that $\forall x \in \mathbb{N}.p(x, y)$.

Since the property holds for all $x \in \mathbb{N}$, it certainly has to hold for a strict subset of $\mathbb{N}$, lets say $A_m = \{x | x = m \vee\ x = s(m) \vee\ x = s(s(m))\}$ (s(x) is the usual successor function).

We may take an arbitrary set $A_m = \{m, m+1, m+2\}$, hence we make no restriction and use it directly without instanciating m.

For hypothesys $< m, y > \in p$,$< m+1, y > \in p$ and $< m+2, y > \in p$.

Our goal is to show that the following statement $y = f(m) = f(m+1) = f(m+2)$ is false, thus falsifying the entire conjecture.

There are two cases:

$$\text{if } m = 2*k \text{ then } \begin{cases} f(m) = f(2*k) = 2*k/2 & = k \\ f(m+1) = f(2*k+1) = (2*k+1-1)/2 & = k \\ f(m+2) = f(2*k+2) = (2*k+2)/2 & = k+1 \end{cases},$$

which clearly contraddicts the hypothesys.

$$\text{if } m = 2*k+1 \text{then } \begin{cases} f(m) = f(2*k+1) = (2*k+1-1)/2 & = k \\ f(m+1) = f(2*k+2) = (2*k+2)/2 & = k+1 \\ f(m+2) = f(2*k+3) = (2*k+3-1)/2 & = k+1 \end{cases},$$

which clearly contraddicts the hyphotesys.

Therefore there exists no $y \in \mathbb{N}$ with such properties, therefore we can dischard the initial hyphotesis and conclude $\neg\exists y \in \mathbb{N}.\forall x \in \mathbb{N}.p(x, y)$.

1.2 Second Trial.

Let $< x, y > \in p \iff y = s(x)$, so that p may be read as the property relationship of "x is the predecessor of y" or, conversely, "y is the successor of x". The function $s(x)$is total and injective, while the property p is anti-reflexive, anti-symmetric and clearly not transitive [proof by exercise].

1:

The first statement is straightforward, since $\mathbb{N}$ is a well-founded set that may be recursively defined as

$0 \in \mathbb{N}$and $\forall x \in \mathbb{N} \Rightarrow s(x) \in \mathbb{N}$

then it is obvious that $\forall x \in \mathbb{N}$, its successor $s(x)$ belongs to the set of $\mathbb{N}$.

2:

The second statement is obviously wrong, since it violates both the injective property of $s(x)$ and the anti-reflexive and anti-symmetric properties of p. Let's try to sort out a proof:

Let $y \in \mathbb{N}$ be, for absurdum, a chosen number such that $\forall x \in \mathbb{N}.p(x, y)$.

Since $y \in \mathbb{N}$, by the property of $\mathbb{N}$ there exist a (possibly large) finite index $n \in \mathbb{N}$ such that $y = s_n(s_{n-1}(...s_1(0)...))$. Also, since $y \in \mathbb{N}$, there must exist $y' \in \mathbb{N}$ such that $y' = s(y)$, which means $< y, y' > \in p$.

For hypothesys y is such that $\forall x \in \mathbb{N}.p(x, y)$, therefore also$< y', y > \in p$ and $y = s(y')$.

Hence one could deduct $y = s(y') = s(s(y))$, which follows into:

$y = s_n(s_{n-1}(...s_1(0)...)) = s_{n+2}(s_{n+1}(...s_1(0)...)))) = s(s(y))$

This is clearly possible only if $n = n + 2$, but clearly $0 \neq 2$.

Since the only hypothesys made was the existence of a$y \in \mathbb{N}$ with such satisfying the second statement, and it was shown that the property didn't hold for an arbitrarily chosen y, we can discard the claim and deduce $\neg\exists y \in \mathbb{N}.\forall x \in \mathbb{N}.p(x, y)$ is true.

Viger