Computability Assignments
Comprehensive List 2012/13

December 3, 2012

Computability Assignment
Year 2012/13 - Number 1

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More information about assignments at http://disi.unitn.it/∼zunino/teaching/computability/assignments

1 Question

Define a binary property $p(x, y)$ over natural numbers such that we have both

1. $\forall x \in \mathbb{N}.\exists y \in \mathbb{N}.p(x, y)$
2. $\neg\exists y \in \mathbb{N}.\forall x \in \mathbb{N}.p(x, y)$

Provide a definition for p, and a proof for the above claims.

1.1 Answer

Write your answer here.

Computability Assignment
Year 2012/13 - Number 2

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2 Question

Let A, B be two sets. Prove that the properties below are equivalent.

- $A = \emptyset \vee B = \emptyset$
- $A \times B = \emptyset$

2.1 Answer

Write your answer here.

3 Preliminaries

Given an infinite sequence of sets $(A_i)_{i\in\mathbb{N}}$, we define $\bigcup_{i=0}^{\infty} A_i = \bigcup\{A_i \mid i \in \mathbb{N}\}$ and $\bigcup_{i=0}^{k} A_i = \bigcup\{A_i \mid i \in \mathbb{N} \wedge i \leq k\} = A_0 \cup A_1 \cup \cdots \cup A_k$.

4 Question

Assume $(A_i)_{i\in\mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \cdots \subseteq \mathbb{N} \ (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcup_{i=0}^{k} A_i$
2. p_2: for all i, if A_i is infinite, then $A_i = A_{i+1}$
3. p_3: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$
4. p_4: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is finite
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite

4.1 Answer

Write your answer here.

Computability Assignment
Year 2012/13 - Number 3

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5 Question

Let A, B be sets, and let $\mathsf{id}_A, \mathsf{id}_B$ denote the identity functions over A and B respectively. Assume $f \in (A \to B)$ and $g \in (B \to A)$ be functions satisfying $g \circ f = \mathsf{id}_A$ and $f \circ g = \mathsf{id}_B$. Prove that f is a bijection (i.e., injective and surjective).

5.1 Answer

Write your answer here.

6 Question

Let A, B be sets, and let $f \in (A \leftrightarrow B)$ be a bijection. Define a bijection $g \in (\mathcal{P}(A) \leftrightarrow \mathcal{P}(B))$ and prove it is such.

6.1 Answer

Write your answer here.

7 Question

Let A, B be two sets, and let $b \notin B$. Define a bijection f between the set of partial functions $(A \rightsquigarrow B)$ and the set of total functions $(A \to B \cup \{b\})$. Prove that is is such.

7.1 Answer

Write your answer here.

Note.

The exercises below are harder. Feel free to skip them if you find them too hard.

8 Question

Define a bijection $f \in [(\mathcal{P}(A) \times \mathcal{P}(B)) \leftrightarrow \mathcal{P}(A \uplus B)]$. Prove that is is such.

8.1 Answer

Write your answer here.

9 Question

Define a bijection $f \in [((A \uplus B) \to C) \leftrightarrow ((A \to C) \times (B \to C))]$. Prove that is is such.

9.1 Answer

Write your answer here.

10 Question

Define a bijection $f \in [((A \to (B \times C)) \leftrightarrow ((A \to B) \times (A \to C))]$. Prove that is is such.

10.1 Answer

Write your answer here.

Computability Assignment
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11 Preliminaries

A partial function g is said to be a *restriction* of a partial function f , written $g \subseteq f$ iff

$$\forall x \in \mathsf{dom}(g).\ g(x) = f(x)$$

Note: this notation "overloads" the symbol $\subseteq$. Indeed, we shall write $A \subseteq B$ to express a subset relation between two sets, and $g \subseteq f$ to express a restriction relation between two functions.

(From a formal point of view, since we defined functions as set of pairs the two notions coincide: the restriction relation above is equivalent to requiring that $\langle a, b\rangle \in g \implies \langle a, b\rangle \in f$ for all a, b, which indeed states that g is a "subset" of f).

12 Question

Let $\mathcal{F}$ be the set of partial functions $\{f \in (\mathbb{N} \rightsquigarrow \mathbb{N}) | \forall x \in \mathbb{N}.\ f(2 \cdot x) = x\}$.

- Define two distinct partial functions f_1, f_2 which belong to $\mathcal{F}$. (I.e, provide two such examples.)
- Define two distinct partial functions g_1, g_2 which do *not* belong to $\mathcal{F}$. (I.e, provide two such examples.)
- Define a partial function $f \in \mathcal{F}$, and consider the set of its *finite* restrictions $\mathcal{G} = \{g \in (\mathbb{N} \rightsquigarrow \mathbb{N}) | g \subseteq f \ \wedge\ \mathsf{dom}(g) \text{ finite}\}$.
  - Define two distinct partial functions h_1, h_2 which belong to $\mathcal{G}$. (I.e, provide two such examples.)
  - Prove whether $\mathcal{F} \cap \mathcal{G} = \emptyset$.

12.1 Answer

Write your answer here.

Note.

The next part is an advanced exercise. I'd suggest to **skip** it, unless you want an extra challenge.

13 Preliminaries

Let $\mathcal{R}$ be a set of inference rules over elements of a set A. Then, $\mathcal{R}$ induces a function $\hat{\mathcal{R}} \in (\mathcal{P}(A) \to \mathcal{P}(A))$ given by

$$\hat{\mathcal{R}}(X) = \{y \mid \exists(\frac{x_1 \ldots x_n}{z}) \in \mathcal{R} \wedge y = z \wedge \forall i \in \{1, \ldots, n\}.x_i \in X\}$$

14 Question

Let m, n range over natural numbers. Consider the following set of inference rules $\mathcal{R}$

$$\frac{n \ m}{n \cdot m} \qquad \frac{}{1} \qquad \frac{n}{n \cdot 2}$$

an the sets

$$E = \{2 \cdot n \mid n \in \mathbb{N}\} \qquad O = \{2 \cdot n + 1 \mid n \in \mathbb{N}\}$$

Then, answer the questions below.

1. State whether $\hat{\mathcal{R}}(O) \subseteq O$
2. State whether $O \subseteq \hat{\mathcal{R}}(O)$
3. State whether $\hat{\mathcal{R}}(E) \subseteq E$
4. State whether $E \subseteq \hat{\mathcal{R}}(E)$
5. State whether $\hat{\mathcal{R}}(\mathbb{N}) \subseteq \mathbb{N}$
6. State whether $\mathbb{N} \subseteq \hat{\mathcal{R}}(\mathbb{N})$
7. State whether $\hat{\mathcal{R}}(E \cup \{1\}) \subseteq E \cup \{1\}$

You may whish to exploit the answer for some question when answering another. Finally:

1. Characterize the minimum fixed point of $\hat{\mathcal{R}}$, i.e. $\bigcap\{X \mid \hat{\mathcal{R}}(X) = X\}$
2. Characterize the maximum fixed point of $\hat{\mathcal{R}}$, i.e. $\bigcup\{X \mid \hat{\mathcal{R}}(X) = X\}$

14.1 Answer

Write your answer here.

Computability Assignment
Year 2012/13 - Number 5

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15 Question

(I am re-proposing this exercise since only a few students solved it. This exercise is rather important, since it involves a reasoning which frequently appeared in past exam questions. While we shall see more examples of these concepts in class, it would be useful to start exercising on that. If you have already sumbitted an answer, skip this and do *not* resubmit your answer please.)

Let $\mathcal{F}$ be the set of partial functions $\{f \in (\mathbb{N} \rightsquigarrow \mathbb{N}) | \forall x \in \mathbb{N}.\ f(2 \cdot x) = x\}$.

- Define two distinct partial functions f_1, f_2 which belong to $\mathcal{F}$. (I.e, provide two such examples.)
- Define two distinct partial functions g_1, g_2 which do *not* belong to $\mathcal{F}$. (I.e, provide two such examples.)
- Define a partial function $f \in \mathcal{F}$, and consider the set of its *finite* restrictions $\mathcal{G} = \{g \in (\mathbb{N} \rightsquigarrow \mathbb{N}) | g \subseteq f\ \wedge\ \mathsf{dom}(g) \text{ finite}\}$.
  - Define two distinct partial functions h_1, h_2 which belong to $\mathcal{G}$. (I.e, provide two such examples.)
  - Prove whether $\mathcal{F} \cap \mathcal{G} = \emptyset$.

15.1 Answer

Write your answer here.

16 Question

Consider the following function:

$$f(n) = \sum_{i=0}^{n} i^2 + i$$

Write a FOR loop implementing f, then translate it in the λ-calculus as program F_1.

Then, write a recursive Java-like function implementing f, and translate it in the λ-calculus as program F_2.

16.1 Answer

Write your answer here.

17 Question

Consider the following function:

$$f(n) = \begin{cases} x^2 + y & \text{if } n = \mathsf{pair}(\mathsf{inL}(x), y) \\ x + 4 \cdot y & \text{if } n = \mathsf{pair}(\mathsf{inR}(x), y) \end{cases}$$

Convince yourself that f is defined for all naturals n, i.e. it is total.

Write a λ-term implementing function f, exploiting the programs $Pair, Proj1, Proj2, InL, InR, Case, \ldots$ we saw in class (also defined in the notes).

17.1 Answer

Write your answer here.

Computability Assignment
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18 Question

Write a λ-term M implementing the following specification:

$$M \ulcorner\!\ulcorner n \urcorner\!\urcorner = \ulcorner \lambda x_0 \ldots x_n .\ x_n x_{n-1} \cdots x_0 \urcorner$$

(Note: The notation $\ulcorner\!\ulcorner n \urcorner\!\urcorner$ above stands for the numeral n, while $\ulcorner N \urcorner$ stands for $\ulcorner\!\ulcorner \# N \urcorner\!\urcorner$ – inside LyX it's hard to tell them apart, but will appear correctly in the PDFs)

18.1 Answer

Write your answer here.

19 Question

Write a λ-term M which, when given as input $\ulcorner N \urcorner$, evaluates to $\ulcorner O \urcorner$, where O is obtained from N by replacing every syntactic occurrence of Ω with $\mathbf{I}$.

To the purpose of this exercise, assume $\Omega = (\lambda x_0.x_0 x_0)(\lambda x_0.x_0 x_0)$ and $\mathbf{I} = \lambda x_0.x_0$.

For example, here are some expected outputs:

$M \ulcorner \lambda x_5.\Omega \urcorner =_{\beta\eta} \ulcorner \lambda x_5.\mathbf{I} \urcorner$
$M \ulcorner \lambda x_3.\mathbf{K}\Omega \urcorner =_{\beta\eta} \ulcorner \lambda x_3.\mathbf{KI} \urcorner$
$M \ulcorner \lambda x_1.x_1 \Omega (\lambda x_7.x_1 \Omega) \urcorner =_{\beta\eta} \ulcorner \lambda x_1.x_1 \mathbf{I} (\lambda x_7.x_1 \mathbf{I}) \urcorner$
$M \ulcorner (\lambda x_0.x_0 x_0)(\lambda x_0.x_0 x_0) \urcorner =_{\beta\eta} \ulcorner \mathbf{I} \urcorner$
$M \ulcorner (\lambda x_1.x_1 x_1)(\lambda x_1.x_1 x_1) \urcorner =_{\beta\eta} \ulcorner (\lambda x_1.x_1 x_1)(\lambda x_1.x_1 x_1) \urcorner$

Hint: use **Sd**, etc. as approprate.

19.1 Answer

Write your answer here.

Computability Assignment
Year 2012/13 - Number 7

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20 Question

Prove that the following set is not λ-definable.

$$A = \{\#M \mid M \text{ has a } \beta\text{-normal form}\}$$

(Hint: show that, if A were λ-definable, then also K_λ would be λ-definable, hence obtaining a contradiction.)

20.1 Answer

Write your answer here.

21 Question

Let A be a λ-definable set. Prove that

$$B = (A \cup \{b_1, \ldots, b_n\}) \setminus \{c_1, \cdots, c_m\}$$

is also λ-definable.

(Hint: do not reinvent the results we saw in class, just apply them.)

21.1 Answer

Write your answer here.

22 Question

Let A be a **non** λ-definable set. Prove that

$$B = (A \cup \{b_1, \ldots, b_n\}) \setminus \{c_1, \cdots, c_m\}$$

is also **non** λ-definable.

(Hint: prove the contrapositive. That is, prove that if B were λ-definable, then also A would be such.)

22.1 Answer

Write your answer here.

Computability Assignment
Year 2012/13 - Number 8

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23 Question

Prove that the following set is **not** λ-definable.

$$A = \{\#M \mid \exists n \in \mathbb{N}.\ M\ulcorner n \urcorner =_{\beta\eta} \ulcorner 5 \urcorner\}$$

23.1 Answer

Write your answer here.

24 Question

Prove that the following set is semantically closed. Then, prove that it **is** λ-definable.

$$A = \{\#M \mid \forall N \in \Lambda.\ N\,M =_{\beta\eta} \mathbf{I}\}$$

24.1 Answer

Write your answer here.

Note.

The following exercise is harder. Feel free to skip it.

25 Question

Prove **whether** the following set is λ-definable.

$$A = \{\#M \mid M\ulcorner M \urcorner =_{\beta\eta} M\}$$

(Note: there is at least one simple solution to this. You do not need to try huge formulae for this.)

25.1 Answer

Write your answer here.

Computability Assignment
Year 2012/13 - Number 9

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26 Question

Assume that f is a recursive partial function satisfying the following property:

$$\forall n, m \in \mathbb{N}.\ f(2^n 3^m) = \begin{cases} m+1 & \text{if } \phi_n(m) = 0 \\ 0 & \text{otherwise} \end{cases}$$

(Note that the above makes no guarantees on e.g. what $f(7)$ actually is).
Prove that:

1. The set $A = \{n \mid \phi_n(5 \cdot n) = 0\}$ is recursive.
2. (Harder, feel free to skip it)
   The set $B = \{n \mid \phi_n(2) = 5\}$ is recursive.

(Note: actually f is a non recursive function, and A, B are non recursive sets. Still, I'm interested in how one proves the above portion of a reduction argument.)

26.1 Answer

Write your answer here.

Computability Assignment
Year 2012/13 - Number 10

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27 Question

Prove that the set

$$A = \{n \mid \phi_n(5+n) = 7\}$$

is $\mathcal{RE}$.

27.1 Answer

Write your answer here.

28 Question

Prove that the set A defined above is **not** recursive, following the sketch below:

1. Prove that $g(n,x) = 7 \cdot \tilde{\chi}_{\mathsf{K}}(n)$ is a recursive partial function.
2. Prove that $f(n) = \#\left(\lambda x. \begin{cases} 7 & \text{if } n \in \mathsf{K} \\ \mathit{undefined} & \text{otherwise} \end{cases}\right)$ is a recursive total function.
3. Prove that $\chi_{\mathsf{K}}(n) = \chi_A(f(n))$ for all n. (If $n \in \mathsf{K}$ then ... If $n \notin \mathsf{K}$ then ...)
4. Prove that is A were recursive, then the set K would be recursive as well.
5. Conclude that A can not be recursive.

28.1 Answer

Write your answer here.

29 Question

Prove whether the set $\bar{A}$ is $\mathcal{RE}$, with A as defined above.

29.1 Answer

Write your answer here.

Computability Assignment
Year 2012/13 - Number 11

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Note

Remember that $undefined \not> x$ for any natural x.

30 Question

Consider the set

$$A = \{n \mid \forall x \in \mathbb{N}.\ \phi_n(x) > x\}$$

Prove that $\mathsf{K} \leq_m A$.

30.1 Answer

Write your answer here.

31 Question

Prove that $\bar{\mathsf{K}} \leq_m A$, with the above A.

31.1 Answer

Write your answer here.

32 Question

Consider the set

$$B = \{\mathsf{pair}(n, m) \mid \phi_n(0) = \phi_m(0)\}$$

Prove that $\bar{\mathsf{K}} \leq_m B$.

32.1 Answer

Write your answer here.