Computability Final Test — 2012-09-03

Notes.

- Write your name and matriculation number on each of your sheets.
- Solve no more than four (4) exercises. This will be strictly enforced: including more than 4 answers will result in the immediate failure of the test.
- Significantly wrong answers will result in negative scores.
- Always provide a justification for your answers.
- To achieve a score ≥ 27 you have to solve the exercise marked with $\star$ below.

Reminder: when equating results of partial functions (as in $\phi_i(3) = \phi_i(5)$), we mean that either 1) both sides of the equation are defined, and evaluate to the same natural number, or 2) both sides are undefined.

Exercise 1. *State whether* $A = \{n \mid \phi_n(\phi_n(1) + 2) = 3\} \in \mathcal{RE}$.

Solution (sketch). $A \in \mathcal{RE}$, since a semi-verifier is given by

$$S_A = \lambda n.\, \mathbf{Eq}\ (\mathbf{Eval1}\ n\ (\mathbf{Add}\ (\mathbf{Eval1}\ n\ \ulcorner 1 \urcorner)\ \ulcorner 2 \urcorner))\ \ulcorner 3 \urcorner\ \mathbf{I}\ \Omega$$

Note that the above halts exactly when both $\phi_n(1)$ is defined (with result z) and $\phi_n(z + 2)$ is defined, with result 3. $\square$

Exercise 2. *State whether* $A = \{n \mid \exists m \in \mathbb{N}.\ m^2 + m = n\} \in \mathcal{R}$.

Solution (sketch). $A \in \mathcal{R}$: to check whether $n \in A$ it is sufficient to check whether $n = 0^2 + 0$ or $n = 1^2 + 1$ or $\ldots$ or $n = n^2 + n$. This can be easily implemented as a loop, so a verifier exists. Note that larger values of m yield $m^2 + m > n$, so there is no need to check for those. $\square$

Exercise 3. *State whether* $A = \{n \mid \mathsf{ran}(\phi_n) \subseteq \mathsf{K}\} \in \mathcal{RE}$.

Solution (sketch). By Rice-Shapiro ($\Leftarrow$), $A \notin \mathcal{RE}$. The set A is semantically closed: let $\mathcal{F}_A$ be its associated set of functions. Take $g(n) = undefined$ for all n. Such g is a finite function. We have $g \in \mathcal{F}_A$ since $\mathsf{dom}(g) = \emptyset \subseteq \mathsf{K}$. Then, take $f(n) = n$, which is a recursive extension of g. We have $f \notin \mathcal{F}_A$ since $\mathsf{dom}(f) = \mathbb{N} \not\subseteq \mathsf{K}$. This concludes. $\square$

Exercise 4. *Let* $A_1 \subseteq A_2$ *and* $h \subseteq g$. *Define*

$$f_1(n) = \begin{cases} g(n) & \text{if } n \in A_1 \\ h(n) & \text{otherwise} \end{cases} \qquad\qquad f_2(n) = \begin{cases} g(n) & \text{if } n \in A_2 \\ h(n) & \text{otherwise} \end{cases}$$

Can we conclude under the hypotheses above that $f_1 \subseteq f_2$*? Provide a proof or a counterexample.*

Solution (sketch). Yes, we have $f_1 \subseteq f_2$. Assuming that $f_1(n)$ is defined, we prove $f_2(n) = f_1(n)$. We consider the following cases:

- If $n \in A_1$, then $f_1(n) = g(n)$ and $n \in A_2$. Hence by definition $f_2(n) = g(n) = f_1(n)$.
- If $n \notin A_1$, then $f_1(n) = h(n)$. Since $f_1(n)$ is defined, $h(n)$ is defined. Since $h \subseteq g$, then $g(n) = h(n) = f_1(n)$. From this we have $f_2(n) = f_1(n)$: in fact we do not need to check whether $n \in A_2$ since both branches in the definition of f_2 coincide ($g(n) = h(n)$).

$\square$

Exercise 5. *Let*

$$A = \{17 \cdot n \mid \phi_n(3) = 4\} \qquad B = \{17^2 \cdot n \mid \phi_n(3) = 4\}$$

State whether $A \leq_m B$, justifying your answer.

Solution (sketch). We have $A \leq_m B$ using $h(n) = 17 \cdot n$ as reduction. Clearly h is recursive and total.

If $n \in A$, then $n = 17 \cdot k$ for some k where $\phi_k(3) = 4$. Hence, $h(n) = 17 \cdot 17 \cdot k \in B$.

If $n \notin A$, then either (1) n is not a multiple of 17 or (2) $n = 17 \cdot k$ for some k but $\phi_k(3) \neq 4$. Then:

- In case (1), $h(n) = 17 \cdot n$ can not be a multiple of 17^2, hence $h(n) \notin B$.
- In case (2), $h(n) = 17 \cdot 17 \cdot k$ is a multiple of 17^2, but we still have $h(n) \notin B$ since $\phi_k(3) \neq 4$.

(Note that no other k needs to be considered, since $g(n) = 17^2 \cdot n$ is injective). $\square$

Exercise 6. *Let $A = \{n \mid \exists x \in \mathbb{N}.\ \phi_n(x) = \phi_x(n)\}$. Prove that $\bar{\mathsf{K}} \cup A \in \mathcal{R}$.*

Solution (sketch). Since $\phi_n(n) = \phi_n(n)$ for all n, we have that $\exists x.\ \phi_n(x) = \phi_x(n)$ holds for all n, hence $A = \mathbb{N}$, which implies $A \cup \bar{\mathsf{K}} \in \mathcal{R}$. $\square$

Exercise 7. *Let*

$$h(n) = \#\Big(\lambda x. \begin{cases} 1 & \text{if } x = 0 \\ \phi_n(x-1) \cdot x & \text{otherwise} \end{cases}\Big)$$

Write h in an equivalent way using the s function from the s-m-n theorem [10% score]. Then, prove that h is a recursive total function [10% score]. Finally, let a be such that $\phi_a = \phi_{h(a)}$ as stated in the second recursion theorem. What is the result of $\phi_a(100)$ [80% score]?

Solution (sketch). Let g such that $g(n,x) = \begin{cases} 1 & \text{if } x = 0 \\ \phi_n(x-1) \cdot x & \text{otherwise} \end{cases}$. This is clearly recursive (the guard is such, as well as the two branches). Take any i such that $\phi_i = g$. We can then define h as $h(n) = s(i, n)$, using the s from the s-m-n theorem.

h is recursive and total because s is such. More pedantically, h is a composition of s (rec.total), the constant function i (rec.total) and the identity n (rec.total).

Let a such that $\phi_a = \phi_{h(a)}$. We have $\phi_a(100) = \phi_{h(a)}(100) = \phi_{s(i,a)}(100) = \phi_i(a, 100) = \phi_a(99) \cdot 100$. By repeating the steps above we get $\phi_a(100) = \phi_{h(a)}(100) = 1 \cdot \cdots \cdot 99 \cdot 100$. Hence the result is 100!. □

Exercise 8. *Let f be a total recursive function such that, for all $n \in \mathbb{N}$:*

$$f(2 \cdot n) < f(2 \cdot n + 2) \qquad f(2 \cdot n + 1) < f(2 \cdot n + 3)$$

Prove that $\mathsf{ran}(f) \in \mathcal{R}$.

Solution (sketch). The two properties above imply that (1) $f(n) < f(n+2)$ for all n, hence by induction we obtain

$$\forall n \in \mathbb{N}.\ f(n) \geq \lfloor n/2 \rfloor$$

More precisely: the above is clearly true when $n = 0$ or $n = 1$. For larger n, we have $f(n) > f(n-2)$ by (1), and by the inductive hypothesis $f(n-2) \geq \lfloor \frac{n-2}{2} \rfloor = \lfloor \frac{n}{2} \rfloor - 1$. Hence $f(n) > \lfloor \frac{n}{2} \rfloor - 1$ which is equivalent to $f(n) \geq \lfloor \frac{n}{2} \rfloor$.

Having established the above, we can check whether a given natural x belongs to $\mathsf{ran}(f)$ by comparing it to $f(0), f(1), \ldots, f(2 \cdot n + 2)$. Indeed, for m larger than $2 \cdot n + 2$, we have $f(m) \geq n + 1 > n$, hence there's no need to try larger m. This test can easily be implemented in a verifier. □

Exercise 9. ⋆ *Prove that there exists $A \subseteq \mathbb{N}$ such that*

$$\mathsf{K} \subseteq A \quad \wedge \quad A \in \mathcal{R} \quad \wedge \quad \bar{A} \text{ infinite}$$

Solution (sketch). Intentionally omitted. □