Computability Final Test — 2012-06-11

Notes.

- Write your name and matriculation number on each of your sheets.
- Solve no more than four (4) exercises. This will be strictly enforced: including more than 4 answers will result in the immediate failure of the test.
- Significantly wrong answers will result in negative scores.
- Always provide a justification for your answers.
- To achieve higher scores (≥ 27) you have to solve the exercise marked with $\star$ below.

Reminder: when equating results of partial functions (as in $\phi_i(3) = \phi_i(5)$), we mean that either 1) both sides of the equation are defined, and evaluate to the same natural number, or 2) both sides are undefined.

Exercise 1. *Comment on this statement by Mr. Rouge Hareng: is it correct?*

> *Let f be a total recursive function. If h_1 is an m-reduction from A to B, then $h_2(n) = f(h_1(n))$ is an m-reduction from A to*
>
> $$B' = \{f(n) \mid n \in B\}$$

Solution (sketch). It is not correct. Take $A = B = \mathsf{K}$, $h_1(n) = n$ and $f(n) = 0$. Note that $B' = \{0\}$ in this case. Clearly $A \leq_m B$ with reduction h_1, but we do not have $\mathsf{K} = A \leq_m B' = \{0\}$ since the latter is recursive. So $h_2(n) = f(h_1(n)) = 0$ can not be a reduction. □

Exercise 2. *Prove whether $A = \{i \mid \forall x.\ x^2 \notin \mathsf{dom}(\phi_i)\} \in \mathcal{RE}$.*

Solution (sketch). Not $\mathcal{RE}$ by Rice-Shapiro ($\Leftarrow$). Indeed, A is semantically closed (easy to check). Take $g(x) = undefined$ to be the always-undefined function. Since $x^2 \notin \mathsf{dom}(g) = \emptyset$ for all x, we have $g \in \mathcal{F}_A$. Also, $\mathsf{dom}(g)$ is finite. Take the constant function $f(x) = 0$ which is clearly recursive and an extension of g. However, $f \notin \mathcal{F}_A$, since e.g. $f(3^2) = 0$ is defined, hence $3^2 \in \mathsf{dom}(f)$. □

Exercise 3. *Prove whether $A = \{i \mid \forall x.\ \phi_x(i) = \phi_x(i^2 + 10 - 6 \cdot i)\} \in \mathcal{RE}$.*

Solution (sketch). The property

$$\forall x.\ \phi_x(i) = \phi_x(i^2 - 6 \cdot i + 10)$$

is true only when $i = i^2 - 6i + 10$, because:

- If $i = i^2 - 6i + 10$, clearly $\phi_x(i) = \phi_x(i^2 - 6i + 10)$ for any x.
- If $i \neq i^2 - 6i + 10$, by taking x to be an index of the identity function $\mathsf{id}(z) = z$, we have $\phi_x(i) = i \neq i^2 - 6i + 10 = \phi_x(i^2 - 6i + 10)$ providing a counterexample to the property above.

Therefore, $A = \{i \mid i = i^2 - 6i + 10\} = \{i \mid i = 2 \vee i = 5\} = \{2, 5\} \in \mathcal{R} \subseteq \mathcal{RE}$ since it is a finite set. □

Exercise 4. *Prove whether* $A = \{i \mid \forall x.\ \exists y.\ y > x^4 \wedge \phi_i(y) = y^2\} \in \mathcal{RE}$.

Solution (sketch). Not $\mathcal{RE}$ by Rice-Shapiro ($\Rightarrow$). Indeed, A is semantically closed (easy to check). Take $f(x) = x^2$, which is recursive and belongs to $\mathcal{F}_A$. Consider any finite restriction g of f. We then have $g(x) = undefined$ as soon as x is large enough, say $x > k$. This implies $g \notin \mathcal{F}_A$ since otherwise we would have, for some $y > k^4 > k$, that $g(y) = y^2 \neq undefined$. Hence, Rice-Shapiro ($\Rightarrow$) concludes. □

Exercise 5. *Consider the function*

$$f(i,j) = \begin{cases} 1 & \textit{if } \mathsf{dom}(\phi_i) = \mathsf{dom}(\phi_j) = \mathbb{N} \wedge \phi_i = \phi_j \\ 0 & \textit{if } \mathsf{dom}(\phi_i) = \mathsf{dom}(\phi_j) = \mathbb{N} \wedge \phi_i \neq \phi_j \\ undefined & \textit{if } \mathsf{dom}(\phi_i) \neq \mathbb{N} \vee \mathsf{dom}(\phi_j) \neq \mathbb{N} \end{cases}$$

State whether $f \in \mathcal{R}$.
(Hint: consider $g(i) = f(i, a)$ *where* a *is picked suitably.)*

Solution (sketch). We prove $f \notin \mathcal{R}$ by contradiction. Assuming $f \in \mathcal{R}$, let a be such that $\phi_a = \mathsf{id}$. Then, $g(n) = f(n, a)$ is a recursive partial function with domain $\mathsf{dom}(g) = \{i | \mathsf{dom}(\phi_i) = \mathbb{N}\} = Tot$. Hence, $Tot \in \mathcal{RE}$. The last statement is false, as can be verified by Rice-Shapiro ($\Rightarrow$). □

Exercise 6. *Define* $A \in \mathcal{RE} \setminus \mathcal{R}$ *and* $B \in \mathcal{RE} \setminus \mathcal{R}$ *such that* $(A \cap B) \notin \mathcal{RE} \setminus \mathcal{R}$.

Solution (sketch). Let $A = \{2 \cdot i \mid i \in \mathsf{K}\}$ and $B = \{2 \cdot i + 1 \mid i \in \mathsf{K}\}$. They are $\mathcal{RE}$ (construct a semi-verifier), but not recursive (K reduces to both). Then, $(A \cap B) = \emptyset \in \mathcal{R}$ concludes. □

Exercise 7. *Let* f *be a total function defined as*

$$f(n) = \begin{cases} 1 + \lfloor \frac{n}{4} \rfloor & n \in \mathsf{K} \\ 0 & \textit{otherwise} \end{cases}$$

State whether a restriction g *of* f *exists such that* all *these properties hold:*

1. $g \in \mathcal{R}$ *and*
2. $\forall x \in \mathsf{dom}(g).\ \exists y \in \mathsf{dom}(g).\ g(x) < g(y)$

Solution (sketch). A trivial, yet correct, answer is to pick $g(x) = undefined$.

A non trivial g can also be constructed as follows. Take a such that $\phi_a = \mathsf{id}$. Consider then

$$A = \{a, \mathsf{pad}(a), \mathsf{pad}(\mathsf{pad}(a)), \dots\} = \{\mathsf{pad}^n(a) \mid n \in \mathbb{N}\}$$

We have $A \in \mathcal{R}$ since to answer the question "$x \in A$?" it is sufficient to compare x to the *increasing* sequence $a, \mathsf{pad}(a), \dots$ until either x is found or the sequence becomes too large ($> x$) — this requires at most x steps.

Now take

$$g(n) = \begin{cases} 1 + \lfloor \frac{n}{4} \rfloor & n \in A \\ undefined & \text{otherwise} \end{cases}$$

since $A \subseteq \mathsf{K}$ (A only contains indices of the identity function), the above is a restriction of

$$h(n) = \begin{cases} 1 + \lfloor \frac{n}{4} \rfloor & n \in \mathsf{K} \\ undefined & \text{otherwise} \end{cases}$$

which is in turn a restriction of f.

We now check properties 1,2:

1) By definition $g \in \mathcal{R}$, since the guard $n \in A$ is recursive, and both branches are recursive functions.

2) If $x \in \mathsf{dom}(g) = A$, then we can take $y = \mathsf{pad}^4(x)$ so that surely $y \in A$ and also $y \geq x + 4$. Hence,

$$g(x) = 1 + \left\lfloor \frac{x}{4} \right\rfloor < 1 + \left\lfloor \frac{x+4}{4} \right\rfloor \leq 1 + \left\lfloor \frac{y}{4} \right\rfloor = g(y)$$

□

Exercise 8. *We write* $\min A$ *for the minimum element of a set* A.

State whether a <u>*partial*</u> *recursive function* f *exists such that,*

$$\forall i \in \mathbb{N}.\ \forall A \subseteq \mathbb{N}.\ (A \neq \emptyset \land \phi_i = \chi_A \implies f(i) = \min A)$$

State whether a <u>*total*</u> *recursive function* g *exists such that,*

$$\forall i \in \mathbb{N}.\ \forall A \subseteq \mathbb{N}.\ (A \neq \emptyset \land \phi_i = \chi_A \implies g(i) = \min A)$$

Solution (sketch). Intentionally omitted. □

Exercise 9. ⋆ *Let* f *be as in Ex. 5. State whether* $g \in \mathcal{R}$ *for some* $g \supseteq f$.

Solution (sketch). There is no such g. (Note in passing that this also proves Ex. 5.)

By contradiction, assume $g \supseteq f$ to be recursive. Let a be such that $\phi_a(x) = 0$. Consider then the function:

$$h(n) = g(\#\left(\lambda x.\ \begin{cases} 1 & \text{if } \phi_n(n) \text{ halts in } x \text{ steps} \\ 0 & \text{otherwise} \end{cases}\right), \#(\lambda x.\ 0))$$

the above h is well defined, since both uses of "$\#(\lambda x. \langle\langle body \rangle\rangle)$" involve recursive bodies w.r.t. n and x. Further, such bodies are also *total* w.r.t. x, i.e. they are always defined. This implies that both indices passed to $g(-,-)$ are indices of total functions, and by hypothesis g coincides with f on those. Function h is also recursive, since g is such. Summing up, we have:

$$
\begin{aligned}
h(n) &= f\left(\#\left(\lambda x.\ \begin{cases} 1 & \text{if } \phi_n(n) \text{ halts in } x \text{ steps} \\ 0 & \text{otherwise} \end{cases}\right), \#(\lambda x.\ 0)\right) \\
&= \begin{cases} 1 & \text{if } \phi_{\#\left(\lambda x.\ \begin{cases} 1 & \text{if } \phi_n(n) \text{ halts in } x \text{ steps} \\ 0 & \text{otherwise} \end{cases}\right)} = \phi_{\#(\lambda x.\ 0)} \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } \left(\lambda x.\ \begin{cases} 1 & \text{if } \phi_n(n) \text{ halts in } x \text{ steps} \\ 0 & \text{otherwise} \end{cases}\right) = (\lambda x.\ 0) \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } \forall x.\ \phi_n(n) \text{ does not halt in } x \text{ steps} \\ 0 & \text{otherwise} \end{cases} \\
&= \chi_{\bar{\mathsf{K}}}(n) \in \mathcal{R}
\end{aligned}
$$

which is a contradiction. □