Computability Final Test — 2011-09-01

Notes.

- Write your name and matriculation number on each of your sheets.
- Solve no more than four (4) exercises. This will be strictly enforced: including more than 4 answers will result in the immediate failure of the test.
- Significantly wrong answers will result in negative scores.
- Always provide a justification for your answers.
- To achieve higher scores (≥ 27) you have to solve the exercise marked with $\star$ below.

Reminder: when equating results of partial functions (as in $\phi_i(3) = \phi_i(5)$), we mean that either 1) both sides of the equation are defined to be the same natural number, or 2) both sides are undefined.

Exercise 1. *If possible, define $f \in \mathcal{R}$ such that the set $\{h \mid h \subseteq f\}$ contains exactly 8 functions; if impossible, explain why. Similarly, if possible define $g \in \mathcal{R}$ such that the set $\{h \mid h \subseteq g\}$ contains exactly 100 functions; if impossible, explain why.*

Solution (sketch). Take $f = \hat{\chi}_{\{0,1,2\}}$, the semi-characteristic of the set $D = \{0, 1, 2\}$. Its domain D contains 3 elements, and the restrictions of f correspond to the subsets of D (i.e. they are of the form $f|_A$ with $A \subseteq D$), so they are $2^3 = 8$. It is impossible to g instead, since 100 is not a power of 2. □

Exercise 2. *State whether $A = \{i \mid \phi_i(\phi_i(22)) = 7\} \in \mathcal{RE}$.*

Solution (sketch). $A \in \mathcal{RE}$. A semi-verifier could simply be

$$S_A = \lambda i.\ \mathbf{Eq}\ \ulcorner 7 \urcorner (\mathbf{Eval}\ i (\mathbf{Eval}\ i\ \ulcorner 22 \urcorner))\mathbf{I}\Omega$$

The above program can loop on the innermost **Eval**, on the outermost, or on Ω. It's easy to consider all these cases and show that indeed S_A halts exactly on A. □

Exercise 3. *Let Even be the set of even naturals.*
Then, state whether $B = \{i \mid \mathsf{ran}(\phi_i) \supseteq Even\} \in \mathcal{RE}$.

Solution (sketch). The set B is semantically closed (because ...). Let f such that $f(n) = n$. We have that $f \in F_B$ since $\mathsf{ran}(f) = \mathbb{N} \supseteq Even$. However, any finite restriction g of f, having finite domain, also has finite range. Hence, we can't have $\mathsf{ran}(g) \supseteq Even$ since $Even$ is infinite. So, we conclude that $B \notin \mathcal{RE}$ by Rice-Shapiro ($\Rightarrow$). $\square$

Exercise 4. *Given*

$$f(x) = \begin{cases} 5 & \text{if } x \text{ is prime} \\ undefined & o.w. \end{cases}$$

state whether $C = \{i \mid \phi_i \supseteq f\} \in \mathcal{RE}$.

Solution (sketch). The set C is semantically closed (because ...). We trivially have that $f \in F_C$ since $f \subseteq f$. However, any finite restriction g of f, having finite domain, can not be defined (as 5) on all the primes which are infinite. Hence, it's not possible that $g \subseteq f$. So, we conclude that $C \notin \mathcal{RE}$ by Rice-Shapiro ($\Rightarrow$). $\square$

Exercise 5. *Let* $A \odot B = (A \setminus B) \cup (B \setminus A)$. *Then state whether each of the following properties holds.*

- *[30% score]* $A, B \in \mathcal{R} \implies A \odot B \in \mathcal{R}$
- *[70% score]* $A \in \mathcal{R} \land B \in \mathcal{RE} \implies A \odot B \in \mathcal{RE}$

Solution (sketch). The first point is true, it's enough to exploit V_A, V_B and add some logical operators to construct $V_{A \odot B}$.

The second point is false, in general. For instance, take $A = \mathbb{N}$ and $B = \mathsf{K}$. Then $A \odot B = (\mathbb{N} \setminus \mathsf{K}) \cup (\mathsf{K} \setminus \mathbb{N}) = \bar{\mathsf{K}} \cup \emptyset = \bar{\mathsf{K}}$ which is not $\mathcal{RE}$. $\square$

Exercise 6. *State whether* $D = \{2^i \mid \forall x.\ \phi_i(x) = \phi_i(2 \cdot x)\} \in \mathcal{RE}$.

Solution (sketch). $D \notin \mathcal{RE}$. Note in passing that D can not be said to be semantically closed, because of the 2^i. We use a simple reduction to cope with that.

First, let $D' = \{i \mid \forall x.\ \phi_i(x) = \phi_i(2 \cdot x)\}$. We have $D' \leq_m D$ with reduction $h(n) = 2^n$ (recursive total because ...). Then, we prove $D' \notin \mathcal{RE}$ by Rice-Shapiro ($\Leftarrow$). D' is semantically closed (because ...). The always undefined function $g(x) = undefined$ belongs to $F_{D'}$, and it is a finite function. So any recursive extension of it should belong to $F_{D'}$ as well, but e.g. $id(x) = x$ does not since for instance for $x = 3$ we have $id(3) \neq id(2 \cdot 3)$. $\square$

Exercise 7. *Let* $f \in (\mathbb{N} \rightsquigarrow \mathbb{N})$ *such that*

$$\forall g \in (\mathbb{N} \rightsquigarrow \mathbb{N}). \Big(g \neq f \land g \subseteq f \implies g \in \mathcal{R} \Big)$$

Prove that $f \in \mathcal{R}$.

Solution (sketch). If f is always undefined then we have $f \in \mathcal{R}$. Otherwise we have $f(x) = y \in \mathbb{N}$ for some x. Let $g(n) = f(n)$ for all $n \neq x$, and $g(x) = undefined$. We have $g \neq f, g \subseteq f$, so by hypothesis $g \in \mathcal{R}$. However, given an implementation for g it's easy to construct an implementation for f, e.g.:

$$F = \lambda n.\ \mathbf{Eq}\ n\ \ulcorner x \urcorner\ \ulcorner y \urcorner\ (G\ n)$$

$\square$

Exercise 8. *Show whether* $\bar{\mathsf{K}} \leq_m \{i \mid \phi_i(i^3 + i^2) = undefined\} = E$.

Solution (sketch). A reduction could be e.g.

$$h(n) = \#(\lambda x.\ \phi_n(n))$$

This is recursive total (because ...). It's simple to check that the above is indeed a reduction, since $\phi_{h(n)}(\langle\text{whatever}\rangle) = \phi_n(n)$ which is undefined iff $n \in \bar{\mathsf{K}}$. $\square$

Exercise 9. *Formally prove that there is some* i *such that* $\phi_i = \chi_{\{5,i,2^i\}}$.

Solution (sketch). Start from

$$f(i,n) = \begin{cases} 1 & \text{if } n = 5 \\ 1 & \text{if } n = i \\ 1 & \text{if } n = 2^i \\ 0 & \text{o.w.} \end{cases}$$

In other words, $f(i,n) = \chi_{\{5,i,2^i\}}(n)$. Such f is clearly recursive, so $f = \phi_x$ for some x. Take function $g(i) = s(x,i)$ where s is from the s-m-n theorem. g is a recursive total function, hence applying the second recursion theorem we get that for some index i, $\phi_i(n) = \phi_{g(i)}(n) = \phi_{s(x,i)}(n) = \phi_x(i,n) = f(i,n) = \chi_{\{5,i,2^i\}}(n)$ which is the desired property.

An alternative, essentially equivalent, way is to start from $h(i) = \#(\lambda n.\ \chi_{\{5,i,2^i\}}(n))$ and apply the second recursion theorem to h. $\square$

Exercise 10. $\star$ *Find* $f \in \mathcal{R}$ *such that*

$$\neg\ (\ \{i \mid \exists x.\ f(x,i) = 0\} \leq_m \{i \mid \forall x.\ f(x,i) = 0\}\ )$$

Solution (sketch). Take

$$f(x,i) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0, i \in \mathsf{K} \\ undefined & \text{o.w.} \end{cases}$$

We have $f \in \mathcal{R}$ (because...). Also, $\{i \mid \forall x.\ f(x,i) = 0\} = \emptyset$ because $f(0,i) = 1$. Also, $\{i \mid \exists x.\ f(x,i) = 0\} = \mathsf{K}$ because $f(x,i) = 0$ only when $i \in \mathsf{K}$ and $x > 0$. So we indeed have $\neg(\mathsf{K} \leq_m \emptyset)$. $\square$