Computability Final Test — 2011-06-07

Notes.

- Write your name and matriculation number on each of your sheets.
- Solve no more than four (4) exercises. This will be strictly enforced: including more than 4 answers will result in immediate failure of the test.
- Significantly wrong answers will result in negative scores.
- Always provide a justification for your answers.
- To achieve higher scores (≥ 27) you have to solve at least one exercise marked with $\star$ below.

Reminder: when equating results of partial functions (as in $\phi_i(3) = \phi_i(5)$), we mean that either 1) both sides of the equation are defined to be the same natural number, or 2) both sides are undefined.

Exercise 1. *Prove that* $\{42\} \leq_m \bar{\mathsf{K}}$.

Solution (sketch). Let a, b be such that $\phi_a(x) = \mathit{undefined}$ and $\phi_b(x) = x$. Then,

$$h(n) = \begin{cases} a & \text{n=42} \\ b & \text{otherwise} \end{cases}$$

is a total recursive function. It is easy to check that it is a reduction (because ...). □

Exercise 2. *Prove or refute each one of the following properties:*

$$\begin{array}{ll} 1) & \forall A. \Big(A \leq_m \bar{\mathsf{K}} \iff A \in \mathcal{RE} \Big) \\ 2) & \forall A. \Big(A \leq_m \bar{\mathsf{K}} \iff \bar{A} \in \mathcal{RE} \Big) \\ 3) & \forall A. \Big(A \leq_m \bar{\mathsf{K}} \iff A \notin \mathcal{RE} \Big) \end{array}$$

Solution (sketch). 1) False, counterexample $A = \bar{\mathsf{K}}$.

2) True: $A \leq_m \bar{\mathsf{K}}$ implies $\bar{A} \leq_m \mathsf{K}$, so $\bar{A} \in \mathcal{RE}$. Similarly, $\bar{A} \in \mathcal{RE}$ implies $\bar{A} \leq_m \mathsf{K}$, hence $A \leq_m \bar{\mathsf{K}}$.

3) False, see Ex.1 and $A = \{42\}$. □

Exercise 3. *Prove whether*

$$B = \{i | \forall x.\ \phi_i(2^x) = x\} \in \mathcal{RE}$$

Solution (sketch). $B \notin \mathcal{RE}$ by Rice-Shapiro ($\Rightarrow$). B is semantically closed (easy to check), so consider the associated set of functions $\mathcal{F}_B = \{\phi_i | i \in B\}$. Take f such that $f(2^n) = n$ and $f(n) = undefined$ if n is not a power of 2. Such f is clearly recursive and belongs to $\mathcal{F}_B$. Consider any finite restriction g of f. Since any such g has a finite domain it can not be defined on all the powers of two, so $g(2^x) = undefined \neq x$ for some x. Hence, any such g can not belong to $\mathcal{F}_B$. □

Exercise 4. *Prove whether*

$$C = \{i | \exists x.\ \phi_i(2^x) = x\} \in \mathcal{RE}$$

Solution (sketch). $C \in \mathcal{RE}$ since it is the range of the following recursive partial function.

$$f(x) = \begin{cases} \mathsf{proj1}(x) & \text{if } \phi_{\mathsf{proj1}(x)}(2^{\mathsf{proj2}(x)}) = \mathsf{proj2}(x) \\ undefined & \text{otherwise} \end{cases}$$

This is recursive (because ...). The range of f is indeed C (because ...). □

Exercise 5. *Prove whether*

$$D = \{i | \forall x < 100.\ \phi_i(x) = 3 \cdot x \ \wedge \ \phi_i(200) = undefined\} \in \mathcal{RE}$$

Solution (sketch). $D \notin \mathcal{RE}$ by Rice-Shapiro ($\Leftarrow$). D is semantically closed (easy to check), so consider the associated set of functions $\mathcal{F}_D = \{\phi_i | i \in D\}$. Take g such that $g(n) = 3 \cdot n$ for all $n < 100$ and undefined otherwise. Such a g is clearly recursive, finite, and belongs to $\mathcal{F}_D$. So, any recursive extension must belong to $\mathcal{F}_D$. However, $f(n) = 3 \cdot n$ for all n is recursive but does not belong to $\mathcal{F}_D$ since $f(200) = 600 \neq undefined$.

□

Exercise 6. *This is an excerpt from a talk by Mr. Rouge Hareng:*

> *I will now prove that the identity function is recursive in a bizarre way. Pick any injective recursive total function f. Then, the identity is the composition of recursive functions $id(x) = f^{-1}(f(x))$, hence it is recursive.*

Comment on Mr. Hareng's proof: is the argument sound? Is the conclusion correct? (Do not forget to provide a justification)

Solution (sketch). The conclusion is obviously correct, since $id \in \mathcal{R}$ by definition of $\mathcal{R}$. The argument is sound, but for the fact that Mr. Hareng provided no justification about why f^{-1} should be recursive. He should have included some algorithm to compute it, e.g.

```
function f_inv(x):
  i := 0 ;
  while f(i) != x do i:=i+1 ;
  return i
```

Note that the above can loop forever if $x \notin \mathsf{ran} f$, but this is not an issue when evaluating $f^{-1}(f(x))$. □

Exercise 7. *Prove that* $\bar{\mathsf{K}} \leq_m E = \{i | \forall x. \exists y.\ y > x \wedge \phi_i(y) = y\}$.

Solution (sketch). Take

$$h(n) = \#\Big(\lambda y. \begin{cases} y & \text{if } \phi_n(n) \text{ does not halt in} \leq y \text{ steps} \\ undefined & \text{otherwise} \end{cases}\Big)$$

The above h is a total recursive function (because ...).

- if $n \in \bar{\mathsf{K}}$, then evaluating $\phi_n(n)$ never halts and

  $$\phi_{h(n)}(y) = \begin{cases} y & \text{if } \phi_n(n) \text{ does not halt in} \leq y \text{ steps} \\ undefined & \text{otherwise} \end{cases} = y$$

  So, for all x, there exists $y = x + 1 > x$ such that $\phi_{h(n)}(y) = y$. Hence $h(n) \in E$.

- if $n \in \mathsf{K}$, then evaluating $\phi_n(n)$ halts, say in k steps. Hence,

  $$\begin{aligned} \phi_{h(n)}(y) &= \begin{cases} y & \text{if } \phi_n(n) \text{ does not halt in} \leq y \text{ steps} \\ undefined & \text{otherwise} \end{cases} \\ &= \begin{cases} y & \text{if } y < k \\ undefined & \text{otherwise} \end{cases} \end{aligned}$$

  Therefore, if we take $x = k$, for all $y > x = k$ we have $\phi_{h(n)}(y) = undefined \neq y$. Hence, $h(n) \notin E$.

□

Exercise 8. *Prove or refute the following property:*

$$\forall f \in (\mathbb{N} \to \mathbb{N}).\ \Big((\forall x.\ f(x) > 2^x) \implies f \in \mathcal{R}\Big)$$

Solution (sketch). False. Take $f(x) = 2^x + 1 + \chi_{\mathsf{K}}(x)$. If this were recursive, then $\chi_{\mathsf{K}}(x) = f(x) - 1 - 2^x$ would be as well. □

Exercise 9. *Let* $A \subseteq \mathbb{N}$ *and* $B = \{x | \exists y.\ \mathsf{pair}(x, y) \in A\} \in \mathcal{RE}$. *Can we conclude that* $A \in \mathcal{R}$ *? If so, provide a proof; otherwise, provide a counterexample.*

Solution (sketch). No, we can not conclude that $A \in \mathcal{R}$. Take for instance $A = \{\mathsf{pair}(x, y) | x \in \mathsf{K} \wedge y \in \mathbb{N}\}$. We have that $B = \mathsf{K} \in \mathcal{RE}$. However $\mathsf{K} \leq_m A$ with reduction $h(n) = \mathsf{pair}(n, 42)$ (easy to check). Hence $A \notin \mathcal{R}$. □

Exercise 10. ⋆ *Given the following assumptions*

$$A \in \mathcal{R} \qquad B \subseteq A \qquad C \subset \bar{A}$$

prove whether $B \leq_m (B \cup C)$.

Solution (sketch). Intentionally omitted. □

Exercise 11. $\star$ *Let $A_0, A_1, \ldots$ be an infinite sequence of sets of natural numbers. Let $f \in \mathcal{R}$ such that*

$$f(x,y) = \tilde{\chi}_{A_x}(y) = \begin{cases} 1 & \text{if } y \in A_x \\ undefined & \text{otherwise} \end{cases}$$

Which of the following conclusions follow from the above? (Remember to provide a justification)

$$\forall x.\ A_x \in \mathcal{RE} \qquad \bigcup_x A_x \in \mathcal{RE} \qquad \bigcap_x A_x \in \mathcal{RE}$$

Solution (sketch). Respectively: true, true, false. Details intentionally omitted. □