Computability Final Test — 2011-02-08

Notes.

- Write your name and matriculation number on each of your sheets.
- Solve only four (4) exercises.
- Significantly wrong answers will result in negative scores.
- Always provide a justification for your answers.
- To achieve higher scores (≥ 27) you have to solve at least one exercise marked with $\star$ below.

Reminder: when equating results of partial functions (as in $\phi_i(3) = \phi_i(5)$), we mean that either 1) both sides of the equation are defined to be the same natural number, or 2) both sides are undefined.

Exercise 1. *This is an excerpt from a talk by Mr. Rouge Hareng:*

> *I will now prove the following fact: if A, B are two non λ-definable sets, then their union $C = A \cup B$ is not λ-definable.*
>
> ***Proof.*** *By contradiction, assume we have a verifier $V_C = \lambda x.\, \mathbf{Or}\,(V_A x)\,(V_B x)$ which λ-defines $A \cup B$. This implies that the verifiers V_A, V_B do exist, contradicting the hypothesis "A, B are not λ-definable".*

Comment on Mr. Hareng's statement and proof. State whether 1) both the statement and proof are correct; or 2) the statement is correct but the proof is not; or 3) the statement is not correct but the proof is; or 4) both the statement and proof are not correct. (Do not forget to provide a justification.)

Solution (sketch). The correct answer is 4) both the statement and the proof are not correct.

Indeed, the statement is false since K and $\bar{\mathsf{K}}$ are not λ-definable sets but their union is $\mathsf{K} \cup \bar{\mathsf{K}} = \mathbb{N}$. Consequently, the proof has to be wrong somewhere. Indeed, whenever $A \cup B$ admits a verifier, it does not follow that the verifier has to be of the form $\lambda x.\, \mathbf{Or}(V_A x)(V_B x)$, so it does not follow that V_A, V_B exist.

Note that answering 3) amounts to claiming that a false statement can have a correct proof! This is the kind of answer that can award negative points. □

Exercise 2. *Show whether $B = \{i + 1 \mid \phi_i(10) = 7\} \in \mathcal{R}$*

Solution (sketch). The set is not recursive. First, it is immediate to verify that

$$B' = \{i|\phi_i(10) = 7\} \leq_m B$$

since $h(n) = n+1$ is a reduction from B' to B. By Rice, B' is not recursive (check the 3 hypotheses). So B is not recursive.

Note: you can not apply Rice directly to B, since it is not semantically closed. □

Exercise 3. *Show that*

$$A = \{i \mid i > 0 \wedge \phi_{i-1}(10) = 7\} \quad \leq_m \quad B = \{i+1 \mid \phi_i(10) = 7\}$$

Solution (sketch). It is easy to check that $A = B$, so $id(n) = n$ is a reduction.

Formally, if $n \in A$, then we have that $n > 0$ and $\phi_{n-1}(10) = 7$. If we let $i = n-1$ we then have $\phi_i(10) = 7$, hence $n = i+1 \in B$.

Dually, if $n \in B$, then $n = i+1$ with $\phi_i(10) = 7$. This implies $n > 0$ and $\phi_{n-1}(10) = \phi_i(10) = 7$, hence $n \in A$.

Note: the above is overly pedantic, a more informal proof that $A = B$ is OK. □

Exercise 4. *Show whether* $C = \{i \mid \phi_i(4) > \phi_i(6)\} \in \mathcal{RE}$

Solution (sketch). $C \in \mathcal{RE}$ since a semi-verifier is provided by $S_C = \lambda x.\mathbf{Gt}(\mathbf{Eval}\ x \ulcorner 4 \urcorner)(\mathbf{Eval}\ x \ulcorner 6 \urcorner)\ I\ \Omega$.

Indeed this S_C stops only whenever both $\phi_x(4)$ and $\phi_x(6)$ are defined, and the former is greater. This is exactly what is required by the definition of C, since undefined values are not greater (or lower) than anything.

□

Exercise 5. *Show whether* $E = \{i \mid \forall x.\ (x^2 < \phi_i(x) < x^3 + 5)\} \in \mathcal{RE}$

Solution (sketch). $E \notin \mathcal{RE}$ by Rice-Shapiro ($\Rightarrow$). The set E is semantically closed (because ...). Let $\mathcal{F}$ be the associated set of functions. Take any function $f \in \mathcal{F}$ (e.g. $f(x) = x^2 + 1$). By Rice-Shapiro ($\Rightarrow$) some finite restriction g of f must belong to $\mathcal{F}$. But this copes with the definition of E, which only allows total functions: g can not be total since it is finite.

□

Exercise 6. *Show whether* $D = \{i \mid \forall x.\ \phi_i(x) = \phi_i(0)\} \in \mathcal{RE}$

Solution (sketch). $D \notin \mathcal{RE}$ by Rice-Shapiro ($\Leftarrow$). The set D is semantically closed (because ...). Let $\mathcal{F}$ be the associated set of functions, and take $f(x) = \mathit{undefined}$ for all x. Clearly, $f(x) = f(0) = \mathit{undefined}$ for all x, so f must belong to $\mathcal{F}$. By Rice-Shapiro ($\Leftarrow$), $\mathcal{F}$ contains any recursive extension of f, e.g. the identity function. Hence, $id(x) = id(0)$ for all x. Therefore $51 = id(51) = id(0) = 0$ which is a contradiction.

□

Exercise 7. *Show that if a generic set $A \in \mathcal{RE}$ then $A \setminus \{42\} \in \mathcal{RE}$*

Solution (sketch). Let $A' = A \setminus \{42\}$. Then $S_{A'} = \lambda x.\, \mathbf{Eq}\, x \ulcorner 42 \urcorner\, \Omega\, (S_A x)$ is a semi-verifier: it diverges on 42 as it should, and otherwise stops only on other $x \in A$.

Alternative solution: The set $B = \{42\}$ is finite, hence recursive. Therefore the complement $\bar{B}$ is recursive, hence $\mathcal{RE}$. Therefore $A' = A \setminus \{42\} = A \setminus B = A \cap \bar{B}$ is $\mathcal{RE}$ since it is the intersection of two $\mathcal{RE}$ sets. □

Exercise 8. *Let $f \in (\mathbb{N} \rightsquigarrow \mathbb{N})$ be a partial function, and let $A_f = \{i \mid \phi_i = f\}$. Discuss whether A_f is an infinite set depending on f.*

Solution (sketch). If f is recursive, by the Padding Lemma, it admits an infinite number of implementations, each one having its own index i, so the set A_f is infinite.

Otherwise, if f is not recursive, no implementation exists and $A_f = \emptyset$ is finite. □

Exercise 9. *Show that $\bar{\mathsf{K}} \leq_m D = \{i \mid \forall x.\, \phi_i(x) = \phi_i(0)\}$*

Solution (sketch). Let

$$h(n) = \#\left(\lambda x. \begin{cases} undefined & x = 0 \\ \phi_n(n) & o.w. \end{cases}\right)$$

Such h is recursive and total (the body of the lambda is recursive w.r.t. the parameters n, x, so we conclude by the s-m-n th.).

Let's check it is a reduction:

- If $n \in \bar{\mathsf{K}}$, $\phi_{h(n)}(x) = \begin{cases} undefined & x = 0 \\ \phi_n(n) & o.w. \end{cases} = \begin{cases} undefined & x = 0 \\ undefined & o.w. \end{cases} = undefined$. Hence $\phi_{h(n)}(x) = \phi_{h(n)}(0)$ for all x, so $h(n) \in D$.
- If $n \notin \bar{\mathsf{K}}$, $\phi_{h(n)}(x) = \begin{cases} undefined & x = 0 \\ \phi_n(n) & o.w. \end{cases}$. Hence $\phi_{h(n)}(1) = \phi_n(n) \neq undefined$ since $n \in \mathsf{K}$, while $\phi_{h(n)}(0) = undefined$, so $h(n) \notin D$.

□

Exercise 10. *Let $g(n) = 1$ when n is a prime number, and $g(n) = 0$ otherwise. Is g primitive recursive?*

Solution (sketch). Yes, g is primitive recursive, since a verifier for the set of prime numbers can be defined in the FOR language: we just need to check for potential divisors within $2..x$. □

Exercise 11. *Construct two sets* $A, B \subseteq \mathbb{N}$ *such that* **all** *of the following properties hold:*

- A *is infinite*
- $A \subset B$ *(note: this implies* $A \neq B$*)*
- $A \in \mathcal{R} \land B \in \mathcal{R}$
- $\forall C. \left(A \subseteq C \subseteq B \implies C \in \mathcal{R}\right)$

Solution (sketch). Take e.g. $A = \mathbb{N} \setminus \{0\}$ and $B = \mathbb{N}$. There are no C's in between but for $C = A$ and $C = B$. So the statement trivially holds. □

Exercise 12. ⋆ *We defined the set of partial functions* $\mathcal{R}$ *as a subset of the set of all the partial functions* $\mathbb{N}^k \rightsquigarrow \mathbb{N}$*. Would it be possible to extend the definition of* $\mathcal{R}$ *in a meaningful way to other cases such as* $\mathbb{Z}^k \rightsquigarrow \mathbb{Z}$*, or* $\mathbb{Q}^k \rightsquigarrow \mathbb{Q}$*? If so, provide the formal details defining such an extension, and comment on them. Otherwise, provide formal details explaining why such an extension would not be meaningful, and comment on them.*

Solution (sketch). Yes, it can be done. A possible way is to encode relative integers and rational numbers into natural numbers using suitable bijections. Then, a function f from e.g. rationals to rationals is defined to be recursive iff there is a recursive function from naturals to naturals which maps the encoding of a rational x to the encoding of the rational $f(x)$. □

Exercise 13. ⋆ *Show whether all bijective functions* $f \in (\mathbb{N} \leftrightarrow \mathbb{N})$ *are recursive.*

Solution (sketch). Intentionally omitted. □