Computability — 2010-09-03

Important Note: this written exam uses a somewhat different notation from other sessions of mine. This is because on 2009 the Computability course was taught by Prof. Baratella, so I used the notation the students were familiar with at that time. While reading, keep in mind the following:

$$\varphi_x = \phi_x \qquad A^c = \bar{A} = \mathbb{N} \setminus A \qquad W_x = \mathsf{dom}\ \phi_x \qquad E_x = \mathsf{ran}\ \phi_x$$

$$A \oplus B = \{\mathsf{inL}(x) | x \in A\} \cup \{\mathsf{inR}(x) | x \in B\}$$

Note. Write your name and matriculation number on each of your sheets. Always provide a justification for your answers.

Note 2. Read the exercises and pick only a few (~ 4) of them. Do not attempt to solve everything. Prioritize quality over quantity.

Note 3. To achieve higher scores (≥ 26) you have to solve at least two exercises marked with $\star$ below.

Exercise 1. *Prove that* K^c *is not r.e.*

Exercise 2. Scenario: *The website* `speakBlue.com` *allows its users to create discussion forums. Each user can post its comments on the site using a simple markup language, with basic formatting commands (e.g. bold, italic, font size, images). In order to improve the flexibility of the system, the web designer Carl is about to allow forum users to use the JavaScript programming language in their comments. In this way, posts would contain dynamic content: when the web page is downloaded, JavaScript content would then run and dynamically generate HTML to be shown to the user. Carl talks about his idea to Sarah, the quality assurance manager. Sarah reminds Carl that there is a strict policy, by which each* `speakBlue.com` *post should never contain any link to its competitor site* `speakOrange.com`*. She is concerned that such a link could be generated through JavaScript code. Carl reassures her, stating that he will make the server to detect JavaScript code generating these links, and reject them. Sarah is still unconvinced.*

Question: *Comment on the above scenario, relating it with computability theory. Make explicit references to definitions and theorems so to support your argument. Summarize the main points, only: do not exceed 10 lines of handwritten text. (Suggestion: do not spend more than 15 minutes writing this.)*

Exercise 3. *Let A, B be r.e. sets. State whether the ones below are r.e. sets. When that is the case, provide a detailed proof.*

$$A \cup B \qquad A \cap B \qquad A \setminus B$$

Exercise 4. *Let $A = \{x : E_x \text{ finite}\}$. Prove whether:*

- *A is recursive*
- *A is r.e., but not recursive*
- *A is not r.e.*

Then, answer again the question above for $A = \{x : E_x \cup W_x = \mathbb{N}\}$.

Exercise 5. *Let $A = \{x : \varphi_x(0) = 0\}$ and $B = \{x : \varphi_x(\varphi_x(0)) = 0\}$. Use the s-m-n theorem to prove that $B \leq_m A$.*

Exercise 6. *Prove that*

$$\Big(A \leq_m C \wedge B \leq_m C\Big) \iff (A \oplus B) \leq_m C$$

Exercise 7. *Given a set A, define $B_A = \{\varphi_x(x) : x \in A\}$. Prove that:*

- *A is r.e. $\implies$ B_A is r.e.*
- *$A = \mathsf{K} \implies B_A$ is recursive.*

Exercise 8. *Let $\langle -, - \rangle$ denote a total recursive bijection between $\mathbb{N}^2$ and $\mathbb{N}$. Consider the following function*

$$f(n) = \begin{cases} \langle x, 0 \rangle & \text{if } \varphi_x(x) \downarrow \text{ in exactly } y \text{ steps} \\ \langle x, y+1 \rangle & \text{otherwise} \end{cases} \qquad \text{where } \langle x, y \rangle = n$$

Then, answer the following questions. 1) Show whether $f \in \mathcal{R}_t$ or not ; 2) show that f is injective ; 3) show whether its inverse $f^{-1} \in \mathcal{R}$ or not.

Exercise 9. *Let $f \in \mathcal{R}_t^{(1)}$ be a bijection $\mathbb{N} \leftrightarrow \mathbb{N}$. State whether $f^{-1} \in \mathcal{R}$ or not (provide a proof or a counterexample).*

Exercise 10. *Let $f \in \mathcal{R}_t^{(2)}$. State whether the set $B = \{x : \forall y \in \mathbb{N}.\ f(x, y) = y\}$ is r.e.*

Exercise 11. $\star$ *Prove or refute the following statements. Below, A, B are subsets of $\mathbb{N}$.*

- *$A \oplus \mathsf{K} \leq_m A \implies A$ r.e.*
- *A r.e. $\implies A \oplus \mathsf{K} \leq_m A$*
- *$A \subseteq B \wedge B$ recursive $\implies A$ r.e.*

Exercise 12. $\star$ *Define two (not six!) sets* A, B *satisfying* **all** *the following properties, taken together:*

- A, B *are not recursive*
- A, B *are r.e.*
- $(A \setminus B) \cup (B \setminus A)$ *is not r.e.*

Exercise 13. $\star$ *Let* A *be an infinite r.e. set. Prove that there exists some infinite recursive set* $B \subseteq A$.

Exercise 14. $\star$ *Let* $f \in \mathcal{R}_t$. *Prove or refute the following.*

- *If* f *is strictly increasing (i.e.* $f(x) < f(y)$ *whenever* $x < y$*), then the range of* f *is a recursive set.*