Computability Final Test — 2009-09-03

Reminder: write your name, surname, and student number. Letters x, y, m, n, i range over $\mathbb{N}$; $A, B, \ldots$ range over subsets of $\mathbb{N}$; M, N, O range over Λ. Justify your answers.

Part 1

Exercise 1.

1. *Show that the set* $A = \{\#M | MM =_{\beta\eta} M(\lambda x.\, xx)\}$ *is closed under* $\beta\eta$*. Then, apply Rice by 1) checking the other hypotheses (*$\mathbf{K}, \mathbf{I}$ *may be useful here), and 2) stating the conclusion.*
2. *Show that the set* $B = \{\#M | MM =_{\beta\eta} (\lambda x.\, xx)M\}$ *is closed under* $\beta\eta$*. Can we apply Rice here?*
3. *Show that if* V_C *is a verifier for* $C = \{(\#M)^2 | M =_{\beta\eta} \mathbf{I}\}$ *then there is a verifier* V_D *for* $D = \{\#M | M =_{\beta\eta} \mathbf{I}\}$*. Show that, whenever* $x \in D$*, we have* $V_D \ulcorner\ulcorner x \urcorner\urcorner = \mathbf{T}$ *and dually, whenever* $x \notin D$*, then* $V_D \ulcorner\ulcorner x \urcorner\urcorner = \mathbf{F}$*. What can we conclude about* C*?*
4. *Construct* M *such that* $M \ulcorner NO \urcorner = \ulcorner O \urcorner$*, for all closed* N, O*. Then construct* P *such that* $P \ulcorner NO \urcorner = O$*, for all closed* N, O*.*
5. *Does* $\#(MNO) = \#(MN'O)$ *imply* $\#N = \#N'$ *?*
6. *Does* $MNO =_{\beta\eta} MN'O$ *imply* $N =_{\beta\eta} N'$ *?*

Exercise 2. *State whether these sets are* λ*-definable.*

$$\begin{aligned} E &= \{\#M | \mathbf{\Theta} M =_{\beta\eta} (\lambda x.\, x)\} \\ F &= \{\#M | M\mathbf{T} =_{\beta\eta} M\mathbf{F} \ulcorner M \urcorner\} \\ G &= \{2^{\#M} \cdot 3^{\#N} | M =_{\beta\eta} N\} \\ H &= \{\mathsf{pair}(\#M, n) | M \ulcorner\ulcorner 5 \urcorner\urcorner =_{\beta\eta} \ulcorner\ulcorner n \urcorner\urcorner\} \end{aligned}$$

Exercise 3. *Optional: solve this only if time allows.*
Adapt the definition of "A is a λ*-definable set" (Def. 80 in the notes) to define "A is a* λ*-semi-definable set" so that its is equivalent to* $A \in \mathcal{RE}$*. Provide a proof sketch of this fact.*

Part 2

Exercise 4.

1. *Define two sets* A, B *such that* $A \notin \mathcal{R}$*,* $B \in \mathcal{R}$*, but* $A \cup B \in \mathcal{R}$*.*

2. *Apply Rice to the set* $A = \{n | \phi_n(3)$ *is even*$\}$. *Show that it is semantically closed, and define the related set* $\mathcal{F}_A$*, check the hypotheses of Rice, and conclude.*

3. *Show that* $\mathsf{K} \leq_m A$*, where* A *is as above.*

4. *Can we conclude that* $A \in \mathcal{RE}$ *from the result above?*

5. *Prove that if* $f \notin \mathcal{R}$*, then* $\mathsf{dom}(f)$ *is infinite.*

6. *A set* A *is* co-finite *iff* $\mathbb{N} \setminus A$ *is finite. Show that co-finite sets belong to* $\mathcal{RE}$.

7. *If* $\mathsf{dom}(f)$ *is finite, can we conclude* $f \in \mathcal{R}$ *? What if instead* $\mathsf{dom}(f)$ *is co-finite ?*

8. *Consider the set* $B = \{n | \forall x.\, \phi_n(x)$ *either undefined or* $> 300 \cdot x\}$. *Show that it is semantically closed, and define the related set* $\mathcal{F}_B$.

   - *Show that* $\mathcal{F}_B$ *is not empty.*
   - *Try to apply Rice-Shapiro in the* ($\Rightarrow$) *direction: what can we conclude in this way?*
   - *Then, try to apply Rice-Shapiro in the* ($\Leftarrow$) *direction: what can we conclude in this way?*

Exercise 5. *Pick five sets from these. State whether the chosen sets belong to* $\mathcal{R}, \mathcal{RE} \setminus \mathcal{R}$*, or neither.*

$$
\begin{aligned}
A &= \{n | \phi_n(\phi_n(5)) = 4\} \\
B &= \{n | \phi_n(5 + n) = 4\} \\
C &= \{n | \mathsf{dom}(\phi_n) = \{2 \cdot m | m \in \mathbb{N}\}\} \\
D &= \{n | \mathsf{ran}(\phi_n) = \{2 \cdot m | m \in \mathbb{N}\}\} \\
E &= \{2 \cdot n | \mathsf{dom}(\phi_n) = \{2 \cdot m | m < 100\}\} \\
F &= \{n | \mathsf{ran}(\phi_n) = \{2 \cdot m | m < 100\}\} \\
G &= \{n | \mathsf{dom}(\phi_n) \textit{ finite or equal to } \mathbb{N}\} \\
H &= \{n | \forall x.\, \phi_n(x) = \phi_n(x+1) \textit{ (and both defined) }\}
\end{aligned}
$$

Nota Bene. Be clear about how you apply the theorems. E.g. if you want to apply Rice-Shapiro, make it clear whether you are using the ($\Rightarrow$) or the ($\Leftarrow$) direction.

Exercise 6. *Optional: solve this only if time allows.*
Prove whether there exists a total recursive function g *such that*

$$\forall x. \Big(\mathsf{dom}(\phi_x) \subseteq \bar{\mathsf{K}} \implies g(x) \in \bar{\mathsf{K}} \setminus \mathsf{dom}(\phi_x)\Big)$$