Computability Final Test — 2009-07-06

Part 1

Exercise 1. *State whether these sets are λ-definable.*

$$
\begin{aligned}
A &= \{\#M | \mathbf{I}M =_{\beta\eta} \Omega M\} \\
B &= \{\#M | \exists n.\, M =_{\beta\eta} \ulcorner n \urcorner \wedge n < 10\} \\
C &= \{\#M | \forall n.\, n < 10 \implies M =_{\beta\eta} \ulcorner n \urcorner\} \\
D &= \{\#M | \exists N.\, \#M + 1 = \#N \wedge N =_{\beta\eta} \mathbf{K}\} \\
E &= \{\#M | \neg(MM =_{\beta\eta} M)\} \\
F &= \{2 \cdot \#M | M \textit{ unsolvable } \}
\end{aligned}
$$

Answer. (short sketch)
Below, I will apply Rice without checking for closure under $\beta\eta$, but this must be done! Even if it is only a matter of expanding the definition, and replacing M with N where possible, you can not omit this check in your answers.

- A is not λ-definable, by Rice. ($\#\mathbf{I} \notin A, \#(\mathbf{\Theta}\Omega) \in A$)
- B is not λ-definable, by Rice. ($\#\ulcorner 5 \urcorner \in B, \#\ulcorner 20 \urcorner \notin B$)
- C is empty, since $M = \ulcorner 0 \urcorner = \ulcorner 1 \urcorner = \cdots = \ulcorner 9 \urcorner$ is impossible.
- D is not λ-definable. Otherwise, assuming V_D we could build a verifier for $D' = \{\#M | M = \mathbf{K}\}$ which is not λ-definable by Rice. Indeed, $V_{D'} = \lambda x.\, V_D(\mathbf{Pred}\, x)$. Also, note that the set D is **not** closed under $\beta\eta$, while D' is.
- E is not λ-definable, by Rice.
- F is not λ-definable. Otherwise, assuming V_F we could build a verifier for $\{\#M | M \text{ unsolvable}\}$ which is not λ-definable by Rice. Also, note that this set is **not** closed under $\beta\eta$.

Exercise 2. *Define* $\mathbf{Triple}, \mathbf{P1}, \mathbf{P2}, \mathbf{P3}$ *such that for all* M, N, O,

$$
\begin{aligned}
\mathbf{P1}(\mathbf{Triple} MNO) &= M \\
\mathbf{P2}(\mathbf{Triple} MNO) &= N \\
\mathbf{P3}(\mathbf{Triple} MNO) &= O
\end{aligned}
$$

Answer. A possible solution:

$$\begin{aligned}&\mathbf{Triple} = \lambda abc.\, \mathbf{Cons}\, a (\mathbf{Cons}\, b\, c)\\ &\mathbf{P1} = \lambda x.\, \mathbf{Fst}\, x\\ &\mathbf{P2} = \lambda x.\, \mathbf{Fst}\, (\mathbf{Snd}\, x)\\ &\mathbf{P3} = \lambda x.\, \mathbf{Snd}\, (\mathbf{Snd}\, x)\end{aligned}$$

Exercise 3. *Define $M \in \Lambda^0$ such that*

$$\begin{aligned}&M \ulcorner \lambda x_i.\, N \urcorner = \ulcorner \lambda x_i.\, \ulcorner N \urcorner \urcorner\\ &M \ulcorner N O \urcorner = \ulcorner N \ulcorner O \urcorner \urcorner\\ &M \ulcorner x_i \urcorner = \ulcorner\ulcorner i \urcorner\urcorner\end{aligned}$$

Answer.

$$\begin{aligned}&M = \lambda x.\, \mathbf{Case}\, x\, A\, B\\ &A = \lambda y.\, y\\ &B = \lambda x.\, \mathbf{Case}\, x\, C\, D\\ &C = \lambda y.\, \mathbf{App}\, (\mathbf{Proj1}\, y)(\mathbf{Num}(\mathbf{Proj2}\, y))\\ &D = \lambda y.\, \mathbf{InR}\, (\mathbf{InR}\, (\mathbf{Pair}\, (\mathbf{Proj1}\, y)(\mathbf{Num}(\mathbf{Proj2}\, y))))\end{aligned}$$

(2010 note: this is now done using **Sd** in a simpler way.)

Exercise 4. *Optional: solve this only if time allows.*
Prove of refute the following:

$$M =_{\beta\eta} MM \implies M =_{\beta\eta} \mathbf{I}$$

Answer. Falsified by $M = \mathbf{\Theta}\, (\lambda x.\, xx)$, which has no normal form.

Exercise 5. *Prove or refute the following.*
If $A \leq_m B$, there exists an injective total recursive function h such that $\forall x \in \mathbb{N}.\, x \in A \iff h(x) \in B$.

Answer. Falsified by $A = \{1, 2\}$ and $B = \{1\}$. They satisfy the hypothesis (easy), but any reduction between them must satisfy $h(1) = h(2) = 1$, so it is not injective.

Exercise 6. *Pick five sets from these. State whether the chosen sets belong to $\mathcal{R}, \mathcal{RE} \setminus \mathcal{R}$, or neither.*

$$\begin{aligned}&A = \{n | \phi_n(2 \cdot n) \text{ halts } \}\\ &B = \{n | \nexists x.\, \phi_n(x) \text{ is even } \}\\ &C = \{n | \mathsf{dom}(\phi_n) \notin \mathcal{RE}\}\\ &D = \{n | \mathsf{dom}(\phi_n) \in \mathcal{R}\}\\ &E = \{n | h \subseteq \phi_n\} \text{ where } h(0) = h(2) = 1, \text{ and undefined otherwise}\\ &F = \{n | \exists g.\, \mathsf{dom}(g) \text{ finite } \wedge \phi_n \subseteq g\}\\ &G = \{\mathsf{pair}(n, m+1) | \phi_n(m) = \phi_m(n) \wedge \text{ both defined } \}\\ &H = \{\mathsf{proj1}(n) | \phi_{\mathsf{proj2}(n)}(\mathsf{proj1}(n)) \text{ halts}\}\\ &I = \{n | n \in \mathsf{ran}(\phi_n)\}\end{aligned}$$

Answer. (sketch)

- $A \in \mathcal{RE}$ since A $= \{n | \exists k.\, \phi_n(2 \cdot n)$ halts in k steps $\}$. Moreover, $A \notin \mathcal{R}$ since $\mathsf{K} \leq_m A$ with reduction $h(n) = \#(\lambda x.\, \phi_n(n))$.

- B is not $\mathcal{RE}$, since $\mathcal{F}_B$ contains the always undefined function, so by Rice-Shapiro ($\Leftarrow$) it would contain $f(x) = 4$ — contradiction.

- $C = \emptyset$, by the definition of $\mathcal{RE}$, so it is recursive.

- $D \notin \mathcal{RE}$, , since $\mathcal{F}_D$ contains the always undefined function, so by Rice-Shapiro ($\Leftarrow$) it would contain $f(n) = \phi_n(n)$ — contradiction since $\mathsf{dom}(f) = \mathsf{K} \notin \mathcal{R}$.

- $E \notin \mathcal{R}$ by Rice. $E \in \mathcal{RE}$ since a semi-verifier can run $\phi_n(0)$ and $\phi_n(1)$ and check the results against 1, and diverge if they are different.

- $F = \{n | \mathsf{dom}(\phi_n) \text{ finite}\} \notin \mathcal{RE}$ by Rice Shapiro ($\Leftarrow$)

- $G \in \mathcal{RE} \setminus \mathcal{R}$ since $\mathsf{K} \leq_m G$ and we can build a semi-verifier for G.

- $H = \mathbb{N} \in \mathcal{R}$

- $I \in \mathcal{RE} \setminus \mathcal{R}$ since we can build a semi-verifier and $\mathsf{K} \leq I$ with reduction

$$h(n) = \# \left(\lambda x. \begin{cases} x & \text{if } \phi_n(n) \text{ halts} \\ undef & \text{otherwise} \end{cases} \right)$$

Exercise 7. *State whether these functions are recursive:*

$$g(n,k) = \begin{cases} 1 & \textit{if } \phi_n(n) \textit{ halts in } k \textit{ steps} \\ 0 & \textit{otherwise} \end{cases}$$
$$h(n,m) = \begin{cases} 1 & \textit{if } \phi_n = \phi_m \\ 0 & \textit{otherwise} \end{cases}$$

Answer.

- $g \in \mathcal{R}$ since we can run the step-by-step interpreter for k steps, and observe termination within that bound.

- $h \notin \mathcal{R}$, since otherwise we can build a verifier for the set

$$A = \{n \mid \phi_n = \mathsf{id}\}$$

  Indeed $V_A = \lambda n.\, h(n, \#\mathbf{I})$. However, $A \notin \mathcal{R}$ by Rice.

Exercise 8. *Optional: solve this only if time allows.*
Assume f to be a partial function $\mathbb{N}^2 \rightsquigarrow \mathbb{N}$ such that

$$\mathsf{dom}(\phi_n) = \mathsf{dom}(\phi_m) = \mathbb{N} \implies f(n,m) = h(n,m)$$

where h is from Ex. 7. Note that this does not constrain f on the indexes of non-total functions. Under this assumption, can we conclude that $f \in \mathcal{R}$? Can we conclude that $f \notin \mathcal{R}$?

Answer. Define

$$a(n) = f(\#(\mathbf{K}\ulcorner 0 \urcorner), \#(\lambda k.\, g(n,k)))$$

where g is from Ex. 7. Note that a is total, since

$$a(n) = h(\#(\mathbf{K}\ulcorner 0 \urcorner), \#(\lambda k.\, g(n,k)))$$

If $f \in \mathcal{R}$, then $a \in \mathcal{R}$. However, we have that $a(n) = 1$ if and only if $g(n,0) = g(n,1) = g(n,2) = \cdots = 0$, that is iff $n \in \bar{\mathsf{K}}$. This is a contradiction, so we can conclude $f \notin \mathcal{R}$.