Computability Final Test — 2009-01-20

Reminder: write your name, surname, and student number. Letters x, m, n range over $\mathbb{N}$; A, B, C, D range over subsets of $\mathbb{N}$; M, N range over Λ.

Part 1

Exercise 1. *Show that* $f(x, y) = (x+y)^{(y+1)}$ *is* λ*-definable.*

Answer.

$$F = \lambda xy.\, \mathbf{Exp}(\mathbf{Add} xy)(\mathbf{Succ} y)$$
$$\mathbf{Exp} = \lambda xy.\, y(\mathbf{Mul}\, x)\ulcorner\!\ulcorner 1 \urcorner\!\urcorner$$

□

Exercise 2. *Compute* $\#(\lambda x_6 x_2.\, x_2)$.

Answer.

$$\mathsf{inR}(\mathsf{inR}(\mathsf{pair}(6, \mathsf{inR}(\mathsf{inR}(\mathsf{pair}(2, \mathsf{inL}(2))))))) = \ldots$$

□

Exercise 3. *Construct* λ*-terms* $\mathbf{Lambda}, \mathbf{Apply}, \mathbf{Var}, \mathbf{Parse}$ *such that:*

$$\mathbf{Lambda}\ulcorner\!\ulcorner n \urcorner\!\urcorner \ulcorner M \urcorner = \ulcorner \lambda x_n.\, M \urcorner$$
$$\mathbf{Apply}\ulcorner M \urcorner \ulcorner N \urcorner = \ulcorner M N \urcorner$$
$$\mathbf{Var}\ulcorner\!\ulcorner n \urcorner\!\urcorner = \ulcorner x_n \urcorner$$
$$\mathbf{Parse}\ulcorner x_n \urcorner V\, A\, L = V \ulcorner\!\ulcorner n \urcorner\!\urcorner$$
$$\mathbf{Parse}\ulcorner M N \urcorner V\, A\, L = A \ulcorner M \urcorner \ulcorner N \urcorner$$
$$\mathbf{Parse}\ulcorner \lambda x_n.\, M \urcorner V\, A\, L = L \ulcorner\!\ulcorner n \urcorner\!\urcorner \ulcorner M \urcorner$$

for all λ*-terms* M, N, V, A, L *and natural* n.

Answer.

$$\mathbf{Lambda} = \lambda nm.\, \mathbf{InR}(\mathbf{InR}(\mathbf{Pair} nm))$$
$$\mathbf{Apply} = \mathbf{App}$$
$$\mathbf{Var} = \mathbf{InL}$$
$$\mathbf{Parse} = \lambda xval.\, \mathbf{Case}\, x\, v\, (\lambda y.\, A)$$
$$A = \mathbf{Case}\, y(\lambda z.\, a(\mathbf{Proj1} z)(\mathbf{Proj2} z))(\lambda z.\, l(\mathbf{Proj1} z)(\mathbf{Proj2} z))$$

(2010 update: these are now included in the notes as $\mathbf{Var}, \mathbf{App}, \mathbf{Lam}, \mathbf{Sd}$.)

□

Exercise 4. *State whether these sets are λ-definable. Justify your answers.*

$$\begin{array}{l} A = \{\#M \mid M =_{\beta\eta} \mathbf{Cons}\,\mathbf{T}\,\mathbf{F} \;\wedge\; \mathbf{K}M\mathbf{IF} =_{\beta\eta} \mathbf{T}\} \\ B = \{\#M \mid \exists n.\, \ulcorner M \urcorner =_{\beta\eta} \ulcorner\ulcorner 3^n \urcorner\urcorner\} \\ C = \{\#M \mid \exists n.\, M \ulcorner\ulcorner 3^n \urcorner\urcorner =_{\beta\eta} \mathbf{I}\} \\ D = \{\#(MN) \mid MM =_{\beta\eta} N\} \end{array}$$

Answer. (sketch) Set A is empty, since $\mathbf{K}M\mathbf{IF} = M\mathbf{F} = \mathbf{Cons}\,\mathbf{T}\,\mathbf{F}\,\mathbf{F} = \mathbf{F}$, and that is a distinct normal form from $\mathbf{T}$. So, A is λ-definable.

For set B, $\ulcorner M \urcorner =_{\beta\eta} \ulcorner\ulcorner 3^n \urcorner\urcorner$ holds iff $\#M = 3^n$, so the set B is actually equal to $\{3^n | n \in \mathbb{N}\}$. That is clearly λ-definable.

For set C, Rice's theorem applies, so it is not λ-definable.

For set D, assume by contradiction that is λ-definable. If so, $V_{D'} = \lambda n.\, V_D(\mathbf{App} \ulcorner \mathbf{I} \urcorner n)$ is a verifier for $D' = \{\#N | \mathbf{I} =_{\beta\eta} N\}$. Indeed, if $\#N \in D'$, then $N = \mathbf{I}$, and $V_{D'} \ulcorner N \urcorner = V_D \ulcorner \mathbf{I} N \urcorner = \mathbf{T}$ since $\mathbf{II} = N$. Otherwise, if $\#N \notin D'$, then $N \neq \mathbf{I}$ and $V_{D'} \ulcorner N \urcorner = V_D \ulcorner \mathbf{I} N \urcorner = \mathbf{F}$ since $\mathbf{II} \neq N$. We reach a contradiction since D' is not λ-definable, as can be shown by Rice. □

Exercise 5. *Show that, for all $F, G \in \Lambda$, there exist $X, Y \in \Lambda$ such that:*

$$X = F \ulcorner Y \urcorner \qquad Y = G \ulcorner X \urcorner$$

You might want to consider $Z = \mathbf{Cons}\, X\, Y$.

Answer. Consider the following equation:

$$Z = \mathbf{Cons}(F \ulcorner \mathbf{Snd}\, Z \urcorner)(G \ulcorner \mathbf{Fst}\, Z \urcorner)$$

That can be written as:

$$Z = (\lambda z.\, \mathbf{Cons}(F(\mathbf{App} \ulcorner \mathbf{Snd} \urcorner z))(G(\mathbf{App} \ulcorner \mathbf{Fst} \urcorner z))) \ulcorner Z \urcorner$$

By the second recursion theorem, such a Z exists. Using that, define X to be $\mathbf{Fst}\, Z$ and Y to be $\mathbf{Snd}\, Z$. Then, a simple check shows that $X = F \ulcorner Y \urcorner$ and $Y = G \ulcorner X \urcorner$. □

Part 2

Exercise 6. *State whether the following functions belong to $\mathcal{R}$. Justify your answer.*

$$f(n) = \begin{cases} 2 \cdot \phi_n(n) + 1 & \textit{if } \phi_n(n) \textit{ is defined} \\ 6 & \textit{otherwise} \end{cases}$$
$$g(n) = \begin{cases} n^2 + 5 \cdot (5 + 2 \cdot n) & \textit{if } \phi_n(n) \textit{ is defined} \\ n \cdot (n + 10) + 25 & \textit{otherwise} \end{cases}$$
$$h(n) = \begin{cases} \phi_n(5) + 51 & \textit{if } \phi_n(5) \textit{ is defined} \\ 700 & \textit{otherwise} \end{cases}$$

Answer.

For f, we have $f(n) = 6$ iff $n \in \bar{\mathsf{K}}$, because $f(n)$ is odd otherwise. So, if we assume $f \in \mathcal{R}$ we reach a contradiction, since we can use that to build a verifier for K.

For g, we have that $g(n) = (n + 5)^2$ is *all* cases, and that is surely computable.

For h, we have that $h(n) = 700$ iff $\phi_n(5)$ is not defined OR $\phi_n(5)$ is defined to be 649. Therefore, h enables us to decide the set $A = \{n | \phi_n(5) = \text{undefined or } \phi_n(5) = 649\}$. This is a contradiction, since A is not recursive, as can be shown by Rice (simple check). ☐

Exercise 7. *State whether the following sets belong to either $\mathcal{R}$, $\mathcal{RE} \setminus \mathcal{R}$, or they do not belong to $\mathcal{RE}$. Justify your answers.*

$$A = \{n \mid \mathsf{dom}(\phi_n) \cap \mathsf{ran}(\phi_n) \neq \emptyset\}$$
$$B = \{n \mid \mathsf{dom}(\phi_n) \setminus \mathsf{ran}(\phi_n) \neq \emptyset\}$$
$$C = \{n \mid \phi_n \textit{ total} \ \wedge \ \forall x.\, \phi_n(x) = \phi_n(\phi_n(x) + 1)\}$$
$$D = \{\mathsf{pair}(n, m) \mid n \in \mathsf{K} \wedge m \in \bar{\mathsf{K}}\}$$

Answer.

For A, we have

$$A = \{n | \exists xyij.\, \phi_n(x) \text{ halts in } i \text{ steps} \wedge \phi_n(y) \text{ halts in } j \text{ steps, with result } x\}$$

The part under the $\exists xyij$ is a decidable predicate, and thus $A \in \mathcal{RE}$. Finally, Rice shows $A \notin \mathcal{R}$: the always undefined function does not belong to A, the identity does, and A is clearly semantically closed.

For B, Rice-Shapiro shows that $B \notin \mathcal{RE}$. If it were, consider g such that $g(0) = 1$ and is undefined otherwise. Then the indexes of g belong to B. By Rice-Shapiro ($\Leftarrow$), all the indexes of every computable extension of g are in B. We reach a contradiction taking the extension $f(0) = 1, f(1) = 0$ (undefined otherwise).

For C, Rice-Shapiro shows that $C \notin \mathcal{RE}$. If it were, consider f such that $f(x) = 5$ for all x's. Then the indexes of f belong to C. By Rice-Shapiro ($\Rightarrow$), there is some finite restriction g having its indexes in the set. Since g can not be total, we have a contradiction.

For D, we note that $\#\mathbf{I} \in \mathsf{K}$. By contradiction, assume $D \in \mathcal{RE}$. We can then build a semi-verifier for $\bar{\mathsf{K}}$ using $S = \lambda n.\, S_D(\mathbf{Pair}\ulcorner I \urcorner n)$. Indeed, S_D is halting iff $\#I$ belongs to K (which is true) and n belongs to $\bar{\mathsf{K}}$, so S indeed works. This is a contradiction since $\bar{\mathsf{K}} \notin \mathcal{RE}$. ☐

Exercise 8. *Prove or refute the following statements:*

- $A, B \in \mathcal{RE} \implies \{\mathsf{pair}(n, m) \mid n \in A \wedge m \in B\} \in \mathcal{RE}$
- $A, B \notin \mathcal{RE} \implies \{\mathsf{pair}(n, m) \mid n \in A \wedge m \in B\} \notin \mathcal{RE}$

Answer. **(sketch)** The first implication is true: to check whether x belongs to the set, take $\mathsf{proj1}(x)$ and $\mathsf{proj2}(x)$ and apply them to S_A and S_B, respectively. This algorithm halts iff both semi-verifiers halt. A simple check shows that this is indeed the case.

The second implication is true: first note that A is not empty (otherwise would be in $\mathcal{RE}$) and pick some $i \in A$. Then we can repeat the argument for set D above, to conclude that $\{\mathsf{pair}(n,m) \mid n \in A \wedge m \in B\} \notin \mathcal{RE}$. □

Exercise 9. *State whether there exists a total $f \in \mathcal{R}$ such that, for all n,*

$$\phi_{f(n)} = \phi_n \qquad \textit{and} \qquad f(n) > 2^n$$

Justify your answer.

Answer. Define f as follows, using the padding function:

$$\begin{aligned}
&f(n) = g(n, 2^n + 1)\\
&g(n, 0) = n\\
&g(n, x+1) = \mathsf{pad}(g(n, x))
\end{aligned}$$

Then, clearly

$$\phi_n = \phi_{g(n,0)} = \phi_{\mathsf{pad}(g(n,0))} = \phi_{g(n,1)} = \ldots = \phi_{g(n,2^n+1)} = \phi_{f(n)}$$

and

$$n = g(n,0) < \mathsf{pad}(g(n,0)) = g(n,1) < \ldots < g(n, 2^n + 1)$$

The last line implies $f(n) \geq 2^n + 1 + n$. □

Exercise 10. *Prove or refute the following statement:*

$$A \in \mathcal{RE} \implies \exists f \in \mathcal{R}.(A = \mathsf{ran}(f) \ \wedge \ f \textit{ injective})$$

Answer. Let S_A be such that $S_A \ulcorner n \urcorner = \mathbf{I}$ when $n \in A$, and unsolvable otherwise. Then, f can be λ-defined as $F = \lambda n.\, S_A n n$. Indeed, it is easy to check that $f = id|_A$, therefore f is injective and $\mathsf{ran}(f) = A$. □