Axiomatization and completeness of lexicographic products of modal logics

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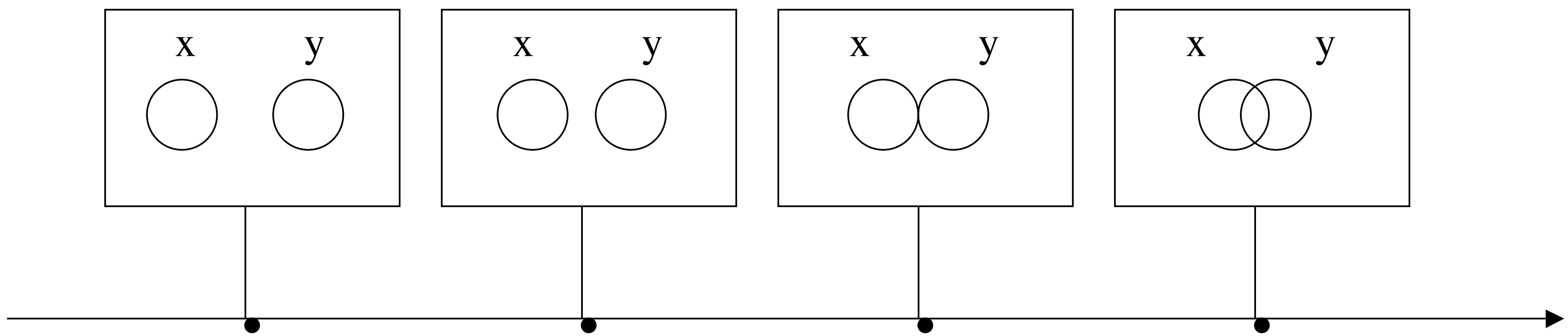
Combining modal logics

- Fusions of modal logics (independent joins)
  - $L_1 \otimes L_2$ is the smallest multimodal logic containing L_1 and L_2
    - $L_1 \otimes L_2$ is a conservative extension of L_1 and L_2 [Thomason, 1980]
    - The fusion operation preserves the following properties
      - Completeness [Kracht and Wolter, 1991]
      - Finite model property [Fine and Schurz, 1996]
      - Decidability [Wolter, 1998]

Combining modal logics

- Spatio-temporal logics
  - ST_0, ST_1 and ST_2 are combinations of BRCC-8 and linear temporal logic
  - $DC(x,y) \rightarrow O(DC(x,y) \vee EC(x,y))$
    - ST_0, ST_1 and ST_2 are decidable [Wolter and Zakharyaschev, 2000]



Combining modal logics

- Description logics with modal operators
  - Extension of concept description languages with modal operators
  - $\langle$john_believes$\rangle\langle$next_year$\rangle$(male_customer$\subseteq\exists$buys.modern_car)
    - [Baader and Ohlbach, 1995]

Combining modal logics

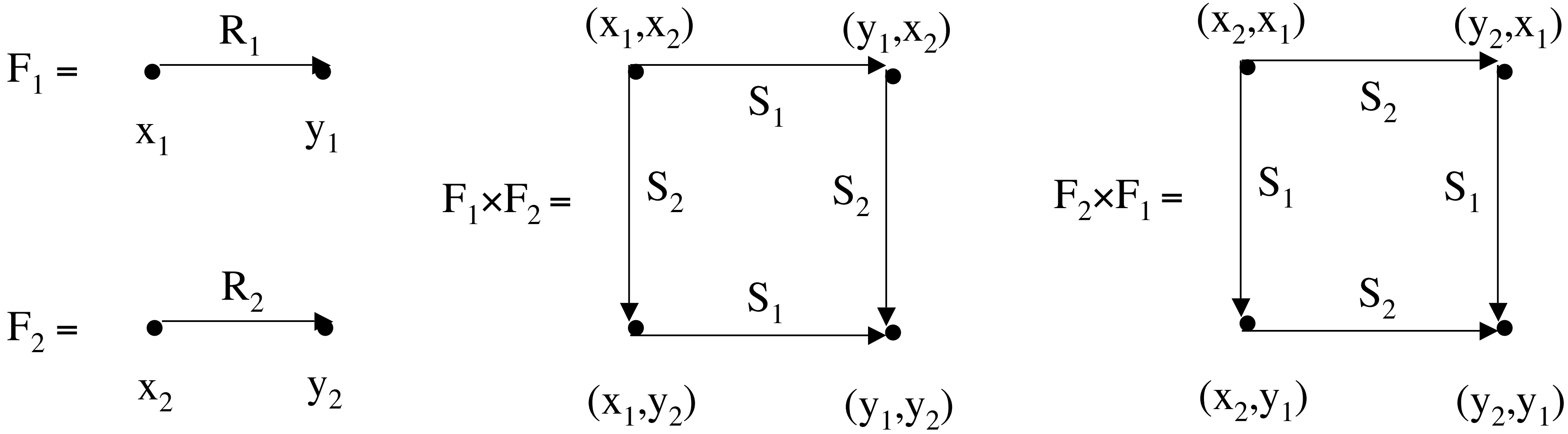
- Products of modal logics
  - Let L_1 and L_2 be Kripke-complete modal logics in $[]_1$ and $[]_2$ respectively
  - $L_1 \times L_2 = Log \{ F_1 \times F_2 : F_1 \models L_1 \text{ and } F_2 \models L_2 \}$
    - [Segerberg 1973]
    - [Shehtman, 1978]
    - [Gabbay and Shehtman, 1998]
    - [Marx, 1999]
    - [Gabbay *et al*, 2003]

Products of relational structures

- Asynchronous products of relational structures
  - $F_1 = (W_1,R_1)$, $F_2 = (W_2,R_2)$
  - $F_1 \times F_2 = (W,S_1,S_2)$ where
    - $W = W_1 \times W_2$
    - $(x_1,x_2)\ S_1\ (y_1,y_2)$ iff $x_1\ R_1\ y_1$ and $x_2 = y_2$
    - $(x_1,x_2)\ S_2\ (y_1,y_2)$ iff $x_1 = y_1$ and $x_2\ R_2\ y_2$

Products of relational structures

- Asynchronous products of relational structures



Products of relational structures

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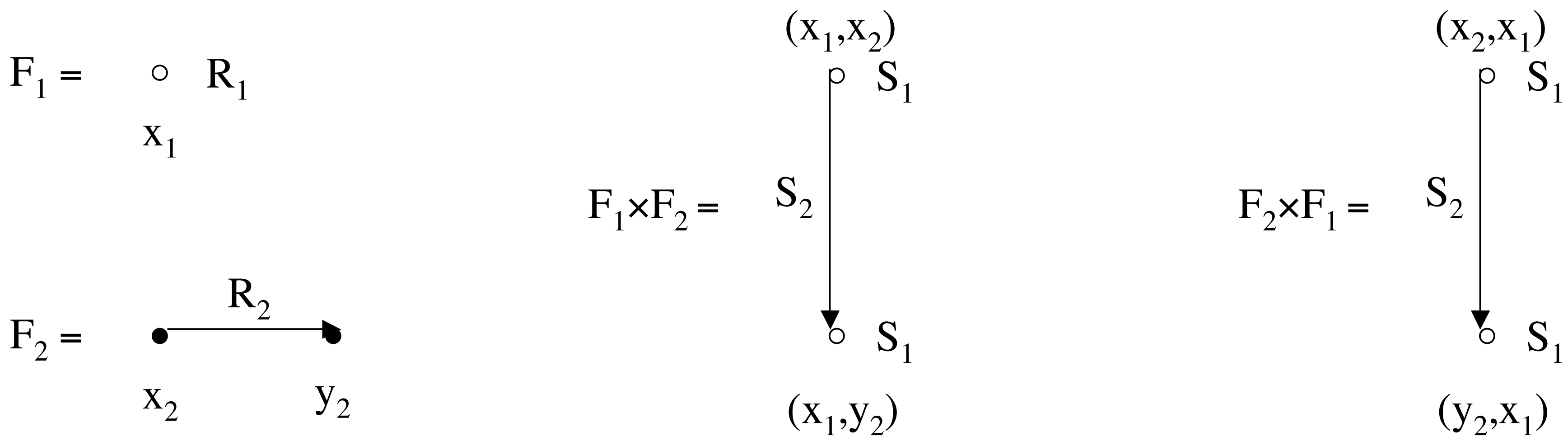

Products of relational structures

- Asynchronous products of linear frames
  - $F_1 = (T_1,<_1)$, $F_2 = (T_2,<_2)$
  - $F_1 \times F_2 = (W,S_1,S_2)$ where
    - $W = T_1 \times T_2$
    - $(x_1,x_2)\ S_1\ (y_1,y_2)$ iff $x_1 <_1 y_1$ and $x_2 = y_2$: « (x_1,x_2) is to the west of (y_1,y_2) »
    - $(x_1,x_2)\ S_2\ (y_1,y_2)$ iff $x_1 = y_1$ and $x_2 <_2 y_2$: « (x_1,x_2) is to the south of (y_1,y_2) »

Products of relational structures

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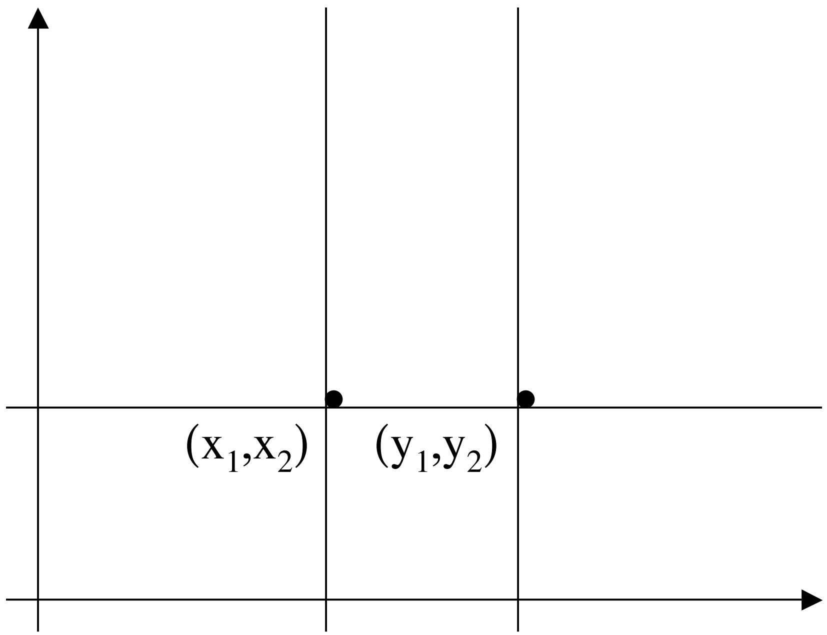

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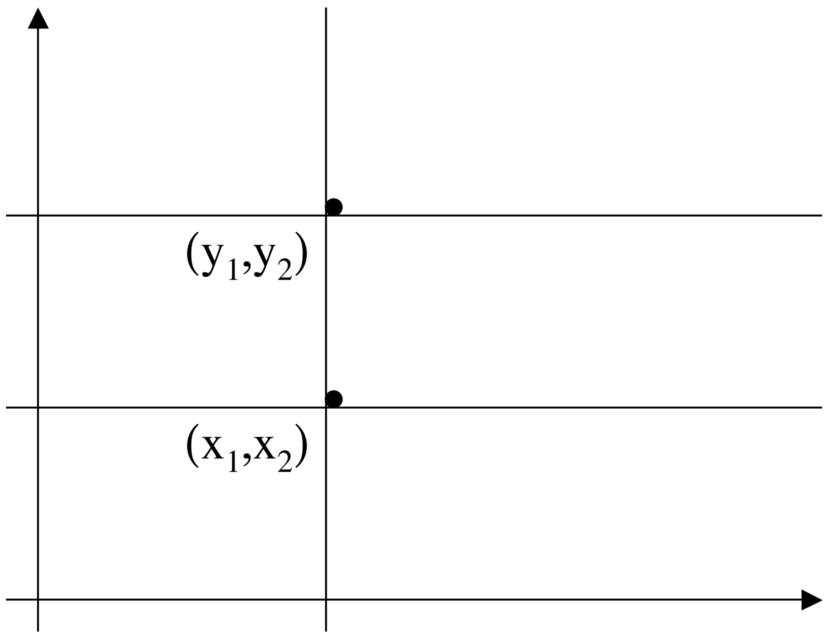

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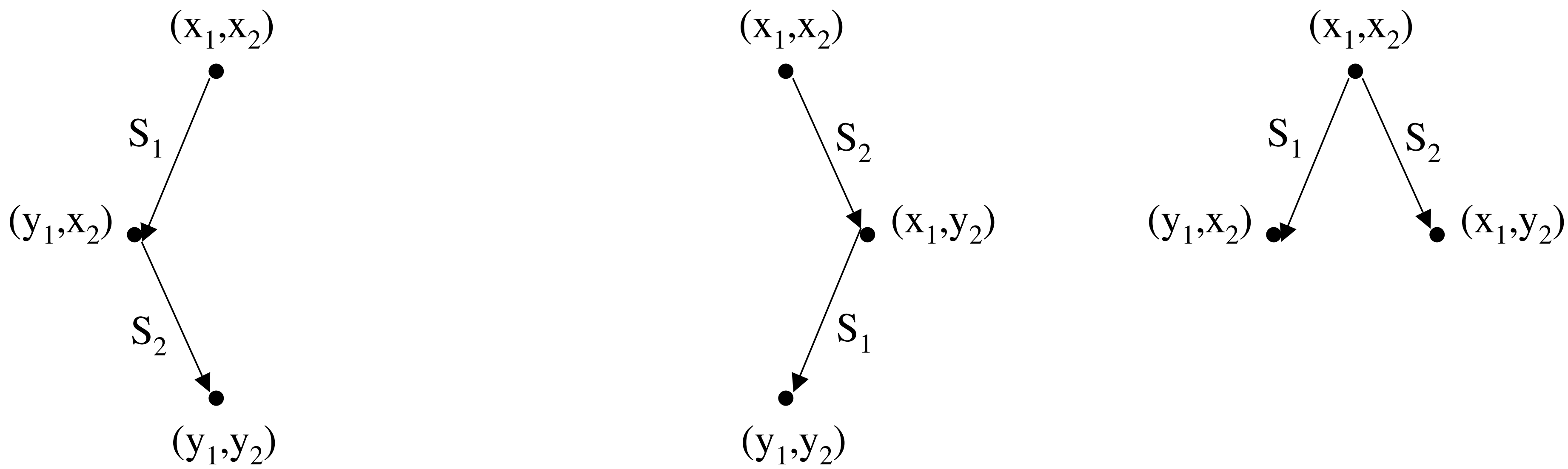

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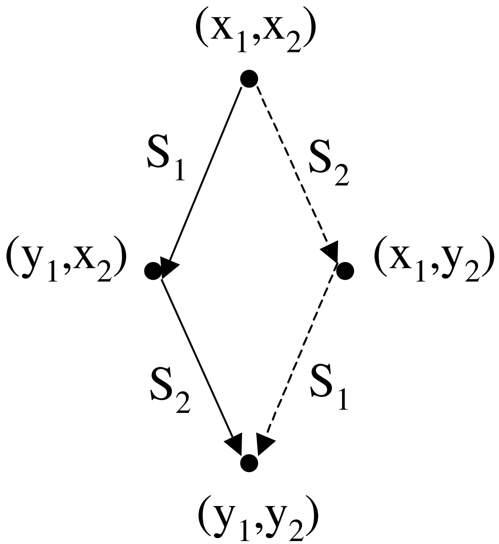
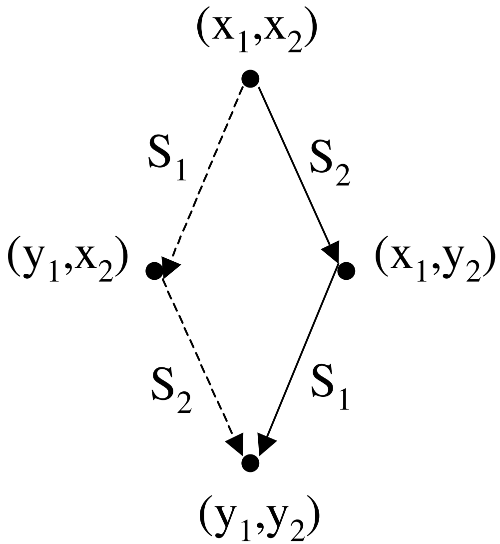
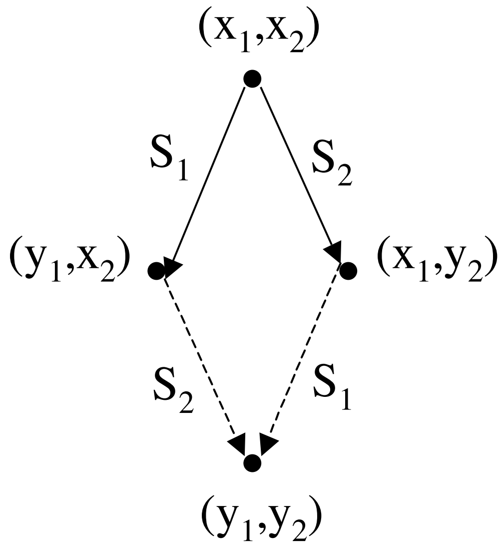






Products of relational structures

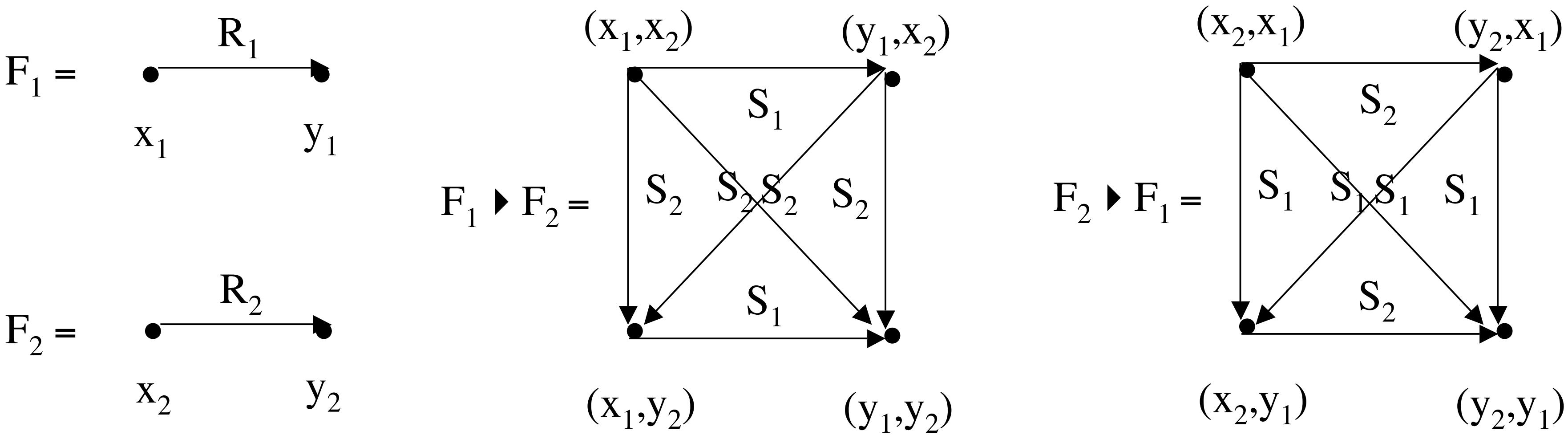
- Asynchronous products of relational structures
  - Let $F = (W,S_1,S_2)$ be countable and such that
    - $\forall x \forall y(\exists z(xS_1z \,\&\, zS_2y) \Rightarrow \exists z(xS_2z \,\&\, zS_1y))$
    - $\forall x \forall y(\exists z(xS_2z \,\&\, zS_1y) \Rightarrow \exists z(xS_1z \,\&\, zS_2y))$
    - $\forall x \forall y(\exists z(zS_1x \,\&\, zS_2y) \Rightarrow \exists z(xS_2z \,\&\, yS_1z))$
  - Then there exists $F_1 = (W_1,R_1)$ and $F_2 = (W_2,R_2)$ such that F is a p-morphic image of $F_1 \times F_2$

Products of relational structures

- Lexicographic products of relational structures
  - $F_1 = (W_1,R_1)$, $F_2 = (W_2,R_2)$
  - $F_1 \blacktriangleright F_2 = (W,S_1,S_2)$ where
    - $W = W_1 \times W_2$
    - $(x_1,x_2)\ S_1\ (y_1,y_2)$ iff $x_1\ R_1\ y_1$ and $x_2 = y_2$
    - $(x_1,x_2)\ S_2\ (y_1,y_2)$ iff $x_2\ R_2\ y_2$

Products of relational structures

- Lexicographic products of relational structures



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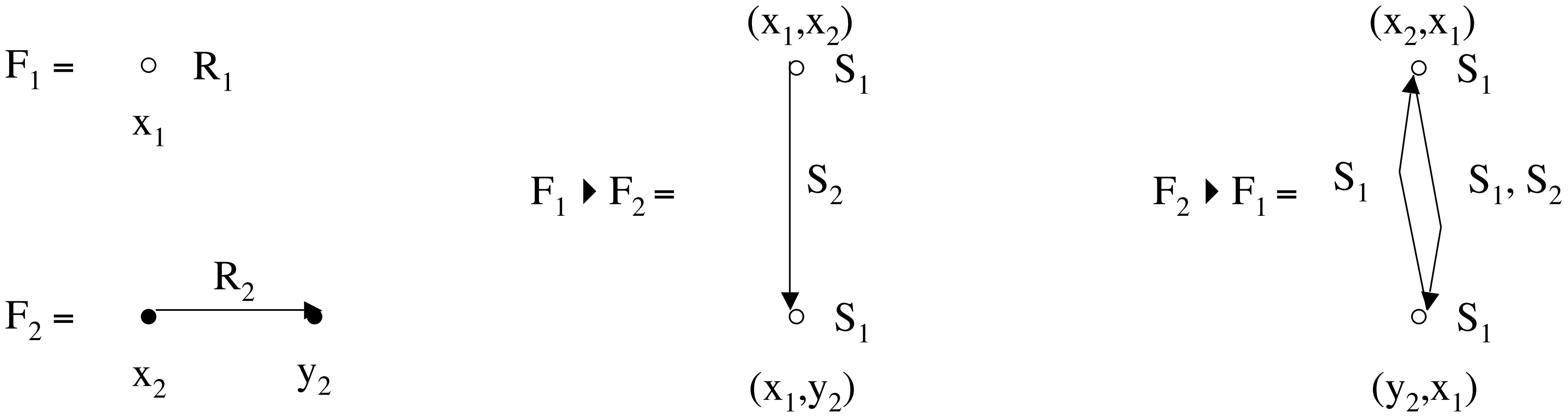

Products of relational structures

- Lexicographic products of linear frames
  - $F_1 = (T_1, <_1)$, $F_2 = (T_2, <_2)$
  - $F_1 \blacktriangleright F_2 = (W, S_1, S_2)$ where
    - $W = T_1 \times T_2$
    - $(x_1,x_2)\ S_1\ (y_1,y_2)$ iff $x_1 <_1 y_1$ and $x_2 = y_2$: « (x_1,x_2) is to the west of (y_1,y_2) »
    - $(x_1,x_2)\ S_2\ (y_1,y_2)$ iff $x_2 <_2 y_2$: « (x_1,x_2) is to the south-west, the south or the south-east of (y_1,y_2) »

Products of relational structures

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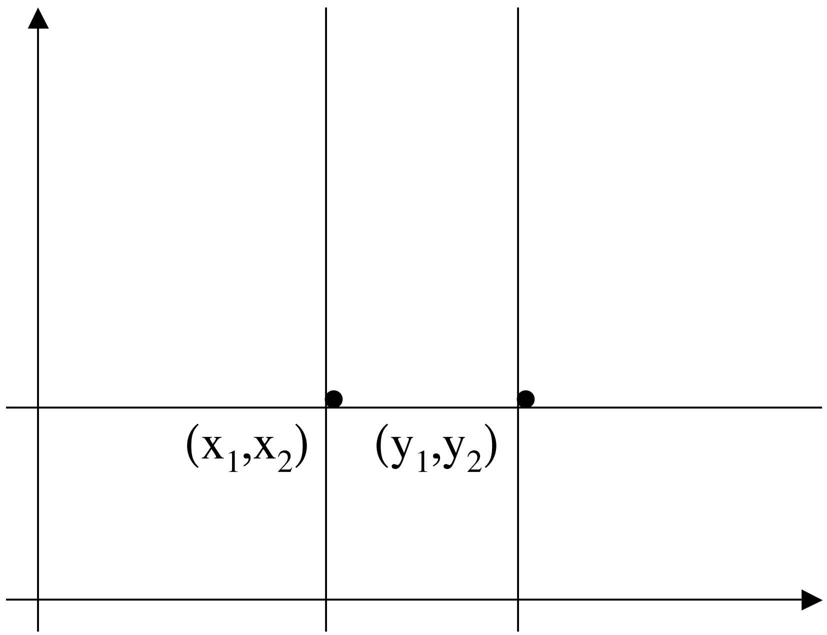

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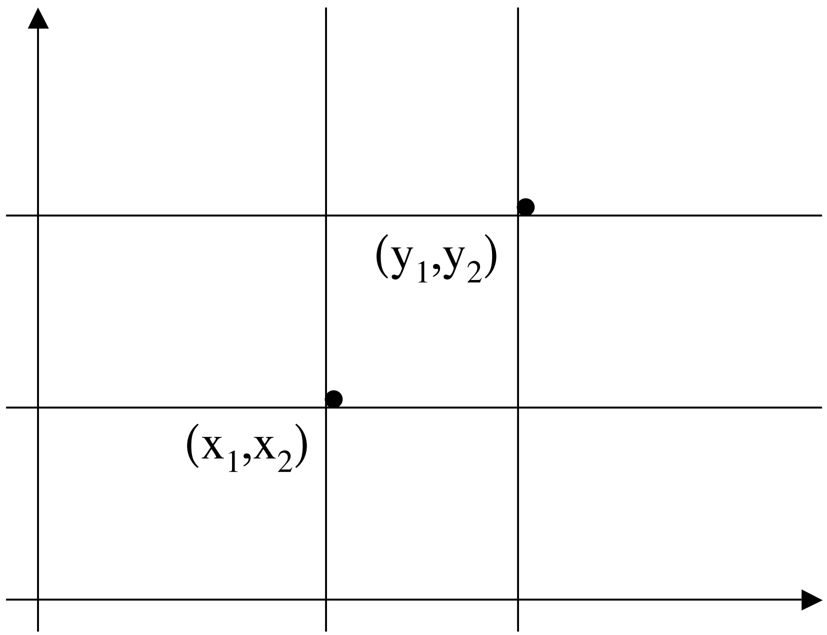

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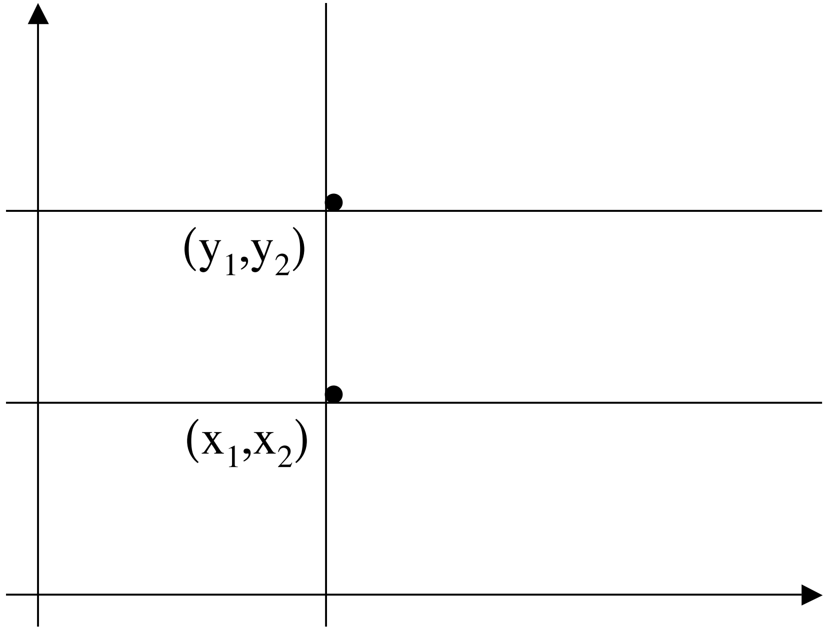

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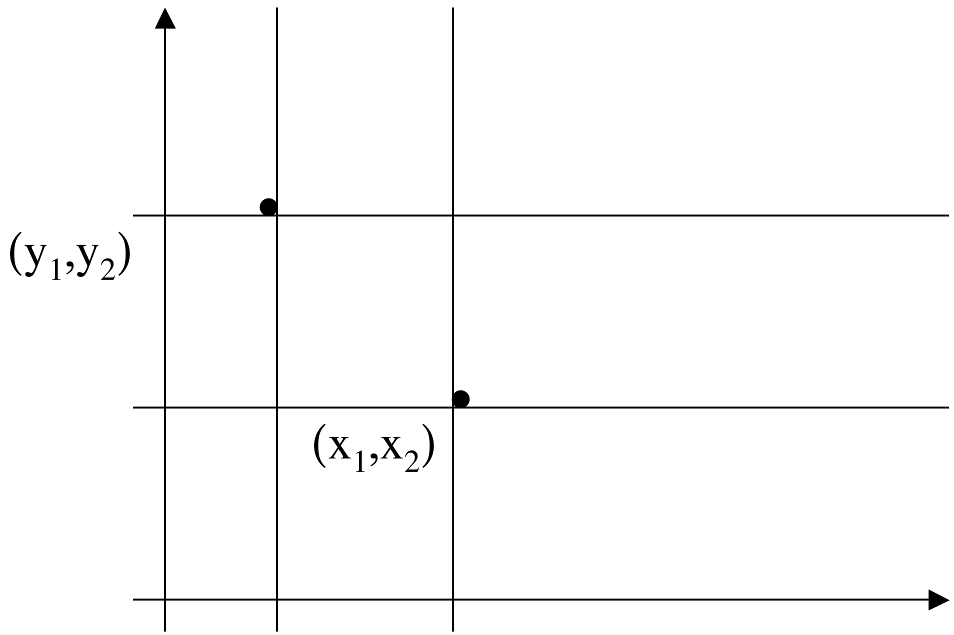

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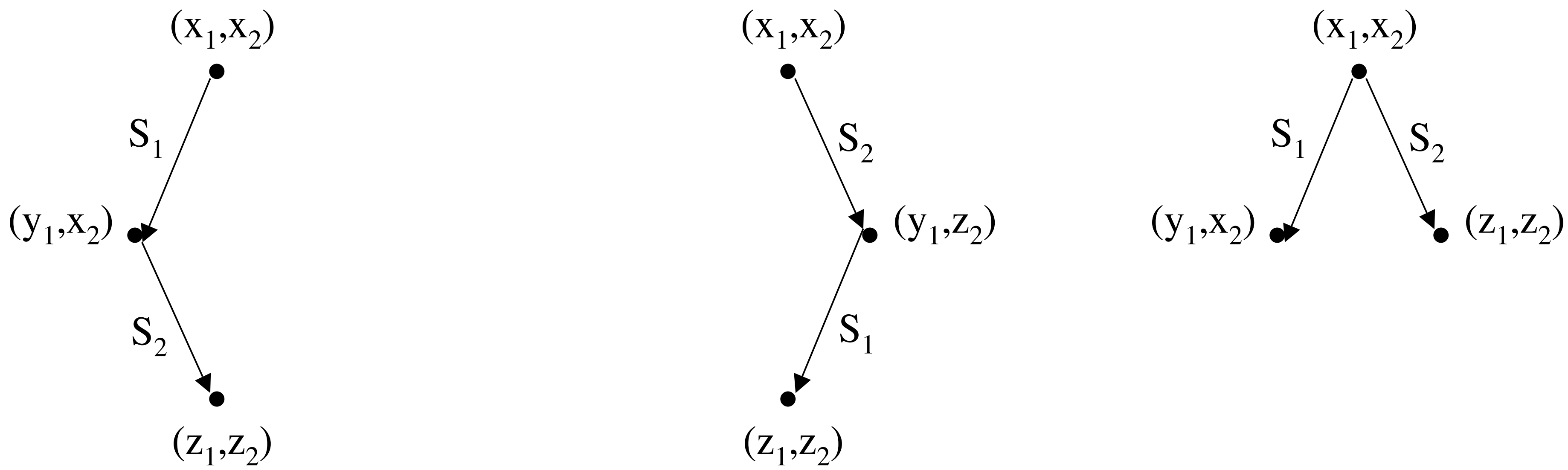

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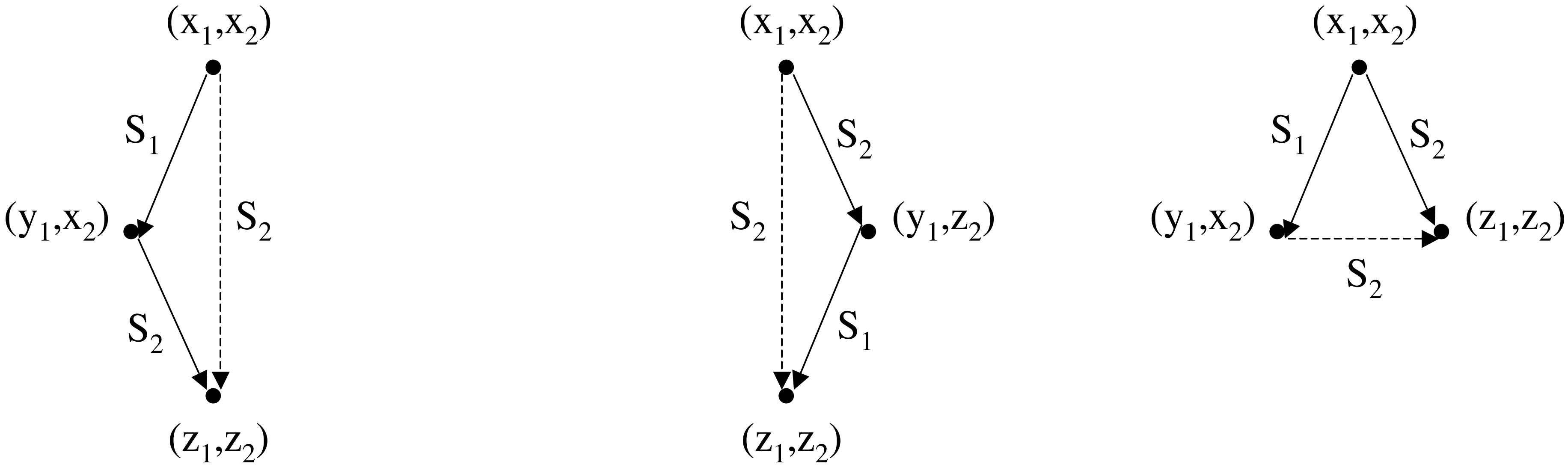
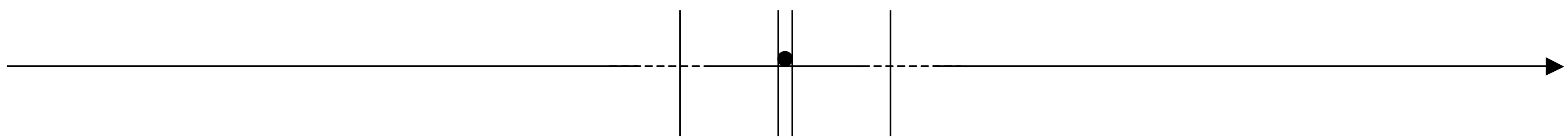


Products of relational structures

- Lexicographic products of relational structures
  - Let $F = (W,S_1,S_2)$ be countable, reflexive and such that
    - $\forall x \forall y(\exists z(xS_1z \ \& \ zS_2y) \Rightarrow xS_2y)$
    - $\forall x \forall y(\exists z(xS_2z \ \& \ zS_1y) \Rightarrow xS_2y)$
    - $\forall x \forall y(\exists z(zS_1x \ \& \ zS_2y) \Rightarrow xS_2y)$
  - Then there exists $F_1 = (W_1,R_1)$ and $F_2 = (W_2,R_2)$ such that F is a p-morphic image of $F_1 \blacktriangleright F_2$

Hyperreals

- Let Hy be the set of all limited hyperreals
- Let $<_{inf}$ and $<_{app}$ be the binary relations on Hy defined by
  - $x <_{inf} y$ iff $x < y$ and $y-x$ is infinitesimal
  - $x <_{app} y$ iff $x < y$ and $y-x$ is appreciable
- The structure $(\text{Hy}, <_{inf}, <_{app})$ is elementary equivalent to the lexicographic of $(\text{Re}, <)$ with itself



Products of modal logics

- Asynchronous products of modal logics
  - Let L_1 and L_2 be Kripke-complete modal logics in $[]_1$ and $[]_2$ respectively
  - $L_1 \times L_2 = \mathrm{Log}\ \{ F_1 \times F_2 : F_1 \models L_1 \text{ and } F_2 \models L_2 \}$
  - $L_1 \times L_2$ is the modal logic in $[]_1$ and $[]_2$ characterized by the class of all frames of the form $F_1 \times F_2$ where $F_1 \models L_1$ and $F_2 \models L_2$

Products of modal logics

- Asynchronous products of modal logics
  - Let L_1 and L_2 be Kripke-complete modal logics in $[]_1$ and $[]_2$ respectively
  - L_1 and L_2 are ×-product matching iff
    - $L_1 \times L_2 = (L_1 \otimes L_2)$ $\oplus\ []_2[]_1 p \rightarrow []_1[]_2 p$
      $\oplus\ []_1[]_2 p \rightarrow []_2[]_1 p$
      $\oplus\ <>_1[]_2 p \rightarrow []_2<>_1 p$

Products of modal logics

- Asynchronous products of modal logics
  - Let L_1 and L_2 be modal logics from the following list : K, D, T, K4, D4, S4, K45, KD45, S5
    - Then L_1 and L_2 are ×-product matching

Products of modal logics

- Asynchronous products of modal logics (examples)
  - $S5 \times S5$ is $(S5 \otimes S5) \oplus []_2[]_1 p \to []_1[]_2 p \oplus []_1[]_2 p \to []_2[]_1 p$
    - Decidable (NEXPTIME-complete)
  - $S4 \times S5$ is $(S4 \otimes S5) \oplus []_2[]_1 p \to []_1[]_2 p \oplus []_1[]_2 p \to []_2[]_1 p$
    - Decidable (NEXPTIME-hard and in N2EXPTIME)
  - $K \times K$ is $(K \otimes K) \oplus []_2[]_1 p \to []_1[]_2 p \oplus []_1[]_2 p \to []_2[]_1 p \oplus <>_1[]_2 p \to []_2<>_1 p$
    - Decidable (NEXPTIME-hard)

Products of modal logics

- Lexicographic products of modal logics
  - Let L_1 and L_2 be Kripke-complete modal logics in $[]_1$ and $[]_2$ respectively
  - $L_1 \blacktriangleright L_2 = \mathrm{Log}\ \{ F_1 \blacktriangleright F_2 : F_1 \models L_1 \text{ and } F_2 \models L_2 \}$
  - $L_1 \blacktriangleright L_2$ is the modal logic in $[]_1$ and $[]_2$ characterized by the class of all frames of the form $F_1 \blacktriangleright F_2$ where $F_1 \models L_1$ and $F_2 \models L_2$

Products of modal logics

- Lexicographic products of modal logics
  - Let L_1 and L_2 be Kripke-complete modal logics in $[]_1$ and $[]_2$ respectively
  - L_1 and L_2 are $\blacktriangleright$-product matching iff
    - $L_1 \blacktriangleright L_2 = (L_1 \otimes L_2) \oplus []_2 p \rightarrow []_1 []_2 p$
      $\oplus []_2 p \rightarrow []_2 []_1 p$
      $\oplus \langle\rangle_1 []_2 p \rightarrow []_2 p$

Products of modal logics

- Lexicographic products of modal logics
  - Let L_1 and L_2 be modal logics from the following list : T, B, S4, S5
    - Then L_1 and L_2 are $\triangleright$-product matching
  - Let L_2 be a modal logic from the following list : K, KB, K4, KB4
    - Then S5 and L_2 are $\triangleright$-product matching
  - Let L_1 be a canonical modal logic
    - Then L_1 and S5 are $\triangleright$-product matching

Products of modal logics

- Lexicographic products of modal logics (examples)
  - $S5 \blacktriangleright S5$ is $(S5 \otimes S5) \oplus []_2 p \rightarrow []_1 p$
    - Decidable (NP-complete)
  - $S4 \blacktriangleright S5$ is $(S4 \otimes S5) \oplus []_2 p \rightarrow []_1 p$
    - Decidable (PSPACE-complete)
  - Is $K \blacktriangleright K$ equal to $(K \otimes K) \oplus []_2 p \rightarrow []_1 []_2 p \oplus []_2 p \rightarrow []_2 []_1 p \oplus <>_1 []_2 p \rightarrow []_2 p \oplus \{ \phi \rightarrow []_{i1} \ldots []_{in} ([]_2 \bot \vee <>_2 \phi) : \phi$ is atom-free and $[]_2$-free $\}$?
    - Decidable ?

Open problems

- Axiomatization/completeness of $K \blacktriangleright K$, $K \blacktriangleright K4$, $K4 \blacktriangleright K$, etc ?
- Finite model property, decidability/complexity of $K \blacktriangleright K$, $K \blacktriangleright K4$, $K4 \blacktriangleright K$, etc
- Transfer theorems for lexicographic products of modal logics comparable with those for fusions of modal logics ?