Formal Methods
Module II: Formal Verification
Ch. 06: **Symbolic Model Checking**

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Outline

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The Need for Fairness Conditions: Intuition

Consider a public restroom. A standard access policy is “first come first served” (e.g., a queue-based protocol).

- Does this policy guarantee that everybody entering the queue will eventually access the restroom?
    - **No**: in principle, somebody might remain in the restroom forever, hindering the access to everybody else
    - In practice, it is considered reasonable to assume that everybody exits the restroom after a finite amount of time

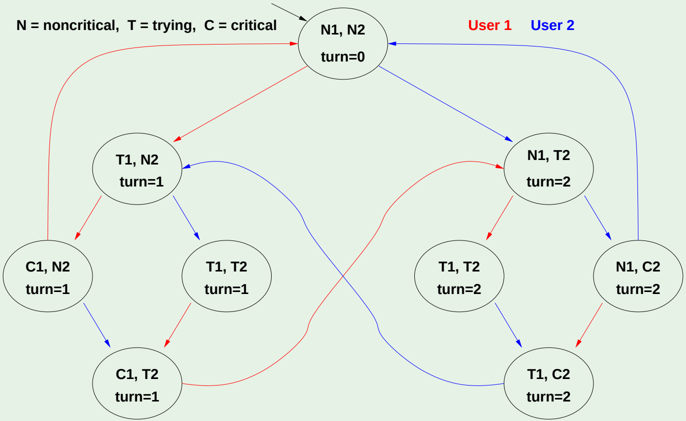
$\Longrightarrow$ It is reasonable enough to assume the protocol suitable under the condition that each user is infinitely often outside the restroom

- Such a condition is called fairness condition

The Need for Fairness Conditions: An Example

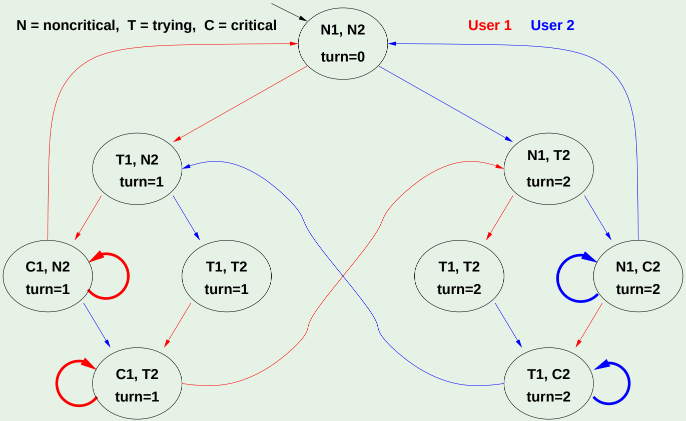
- Consider a variant of the mutual exclusion in which one process can stay permanently in the critical zone
- Do $M \models \mathbf{G}(T_1 \to \mathbf{F}C_1)$, $M \models \mathbf{G}(T_2 \to \mathbf{F}C_2)$ still hold?

The Need for Fairness Conditions: An Example [cont.]



$M \models \mathbf{G}(T_1 \rightarrow \mathbf{F}C_1)$ $\quad$ $M \models \mathbf{G}(T_2 \rightarrow \mathbf{F}C_2)$

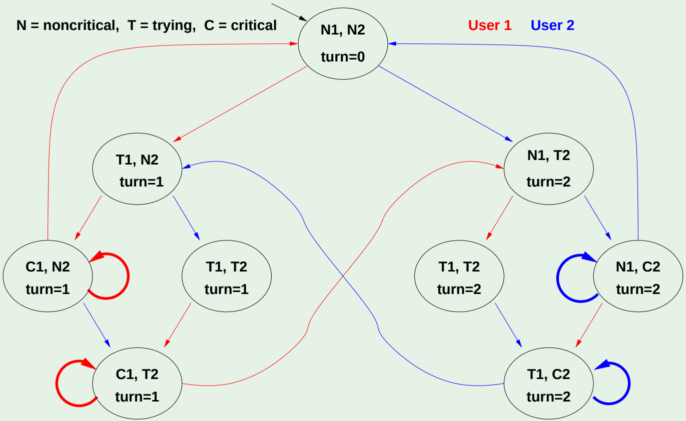
The need for fairness conditions: an example [cont.]



$M \models \mathbf{G}(T_1 \to \mathbf{F}C_1)$?

$M \models \mathbf{G}(T_2 \to \mathbf{F}C_2)$?

The need for fairness conditions: an example [cont.]



$\mathbf{G}(T_1 \rightarrow \mathbf{F}C_1)$? $\quad$ $\mathbf{G}(T_2 \rightarrow \mathbf{F}C_2)$?

NO: E.g., it can cycle forever in $\{C_1, T_2, turn = 1\}$

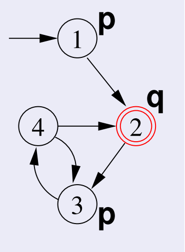
$\Longrightarrow$ Unfair protocol: one process might never be served

Fairness Conditions

- It is desirable that certain (typically Boolean) conditions φ's hold infinitely often: $\mathbf{GF}\varphi$
- $\mathbf{GF}\varphi$ is called fairness condition
- Intuitively, fairness conditions are used to eliminate behaviours in which a certain condition φ never holds:
  $\mathbf{GF}\varphi$: "it is never reached a state from which φ is forever false"
- Example: it is not desirable that, once a process is in the critical section, it never exits: $\mathbf{GF}\neg C_1$
- A fair condition φ_i can be represented also by the set f_i of states where φ_i holds
  $(f_i := \{s : \pi, s \models \varphi_i, \ \textit{for each}\ \pi \in M\})$

Fair Kripke models

- A Fair Kripke model $M_F := \langle S, R, I, AP, L, F \rangle$ consists of:
  - a set of states S;
  - a set of initial states $I \subseteq S$;
  - a set of transitions $R \subseteq S \times S$;
  - a set of atomic propositions AP;
  - a labeling function $L : S \longmapsto 2^{AP}$;
  - a set of fairness conditions $F = \{f_1, \ldots, f_n\}$, with $f_i \subseteq S$.



- E.g., $\{\{2\}\} := \{\{s : L(s) = \{q\}\}\} = \{\mathbf{GF}q\}$ is the set of fairness conditions of the Kripke model above
- Fair path π: at least one state for each f_i occurs infinitely often in π (φ_i holds infinitely often in π: $\pi \models \mathbf{GF}\varphi_i$)
  - E.g., every path visiting infinitely often state 2 is a fair path.
- Fair state: a state through which at least one fair path passes
  - E.g., all states 1,2,3,4 are fair states
- Note: fair state $\neq$ state belonging to a fairness condition

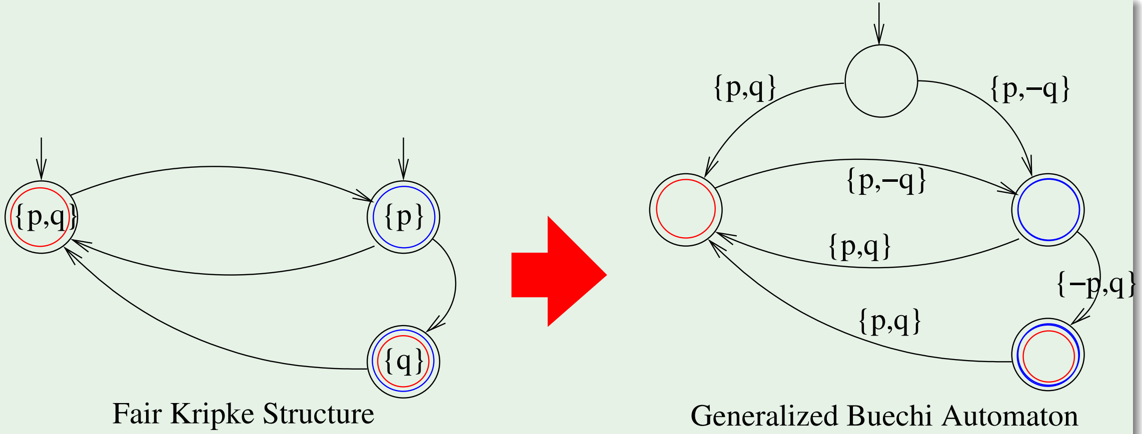
Computing an NBA A_M from a Fair Kripke Model M

- Transforming a fair K.S. $M = \langle S, S_0, R, L, AP, FT\rangle$, $FT = \{F_1, ..., F_n\}$, into a generalized NBA $A_M = \langle Q, \Sigma, \delta, I, FT'\rangle$ s.t.:
  - States: $Q := S \cup \{init\}$, *init* being a new initial state
  - Alphabet: $\Sigma := 2^{AP}$
  - Initial State: $I := \{init\}$
  - Accepting States: $FT' := FT$
  - Transitions:

$$\begin{aligned}\delta : \quad & q \xrightarrow{a} q' \text{ iff } (q, q') \in R \; and \; L(q') = a \\ & init \xrightarrow{a} q \text{ iff } q \in S_0 \; and \; L(q) = a\end{aligned}$$

- $\mathcal{L}(A_M) = \mathcal{L}(M)$
- $|A_M| = |M| + 1$

Computing a (Generalized) BA A_M from a Fair Kripke Structure M: Example



$\Longrightarrow$ Substantially, add one initial state, move labels from states to incoming edges, set fair states as accepting states

Outline

CTL M.C. with Fair Kripke Models

Fair Kripke Models restrict the M.C. process to fair paths:

- $M_f \models \varphi$ iff $\pi \models \varphi$ for every fair path π
- Path quantifiers (from CTL) apply only to fair paths:
  - $M_F, s \models \mathbf{A}\varphi$ iff $\pi, s \models \varphi$ for every fair path π s.t. $s \in \pi$
  - $M_F, s \models \mathbf{E}\varphi$ iff $\pi, s \models \varphi$ for some fair path π s.t. $s \in \pi$

$\Longrightarrow$ a fair state s is a state in M_F iff $M_F, s \models \mathbf{EG}\mathit{true}$.

- We need a procedure to compute the set of fair states: `Check_FairEG(true)`

Example

- $M_f \models \mathbf{EG}\mathit{true}$? yes
- $M_f \models \mathbf{G}(p \rightarrow \mathbf{F}q)$? yes
- $M \models \mathbf{G}(p \rightarrow \mathbf{F}q)$? no

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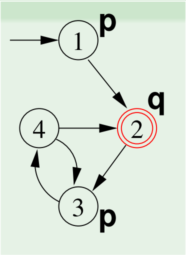

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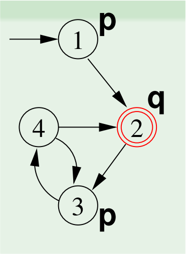

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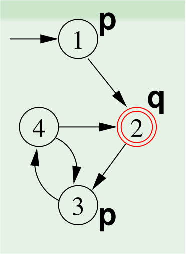

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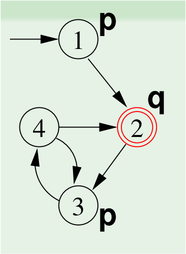

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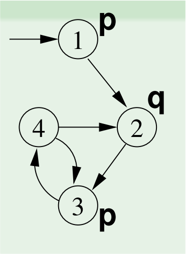

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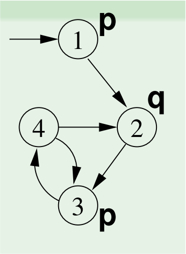
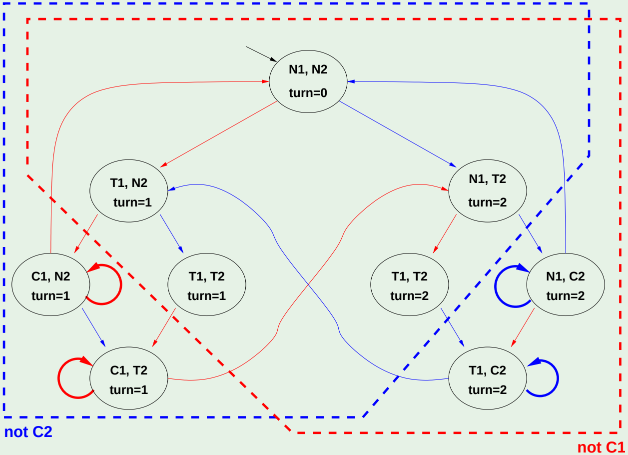


Fair CTL Model Checking: Example

F := {{ not C1},{not C2}}



$M_F \models \mathbf{G}(T_1 \rightarrow \mathbf{F}C_1)$? $M_F \models \mathbf{G}(T_2 \rightarrow \mathbf{F}C_2)$? YES: every fair path satisfies the conditions

CTL M.C. vs. LTL M.C. with Fair Kripke Models

Remark: fair CTL M.C.

When model checking a CTL formula ψ, fairness conditions cannot be encoded into the formula:

$$M_{\{f_1,\ldots,f_n\}} \models \psi \not\Longleftrightarrow M \models (\bigwedge_{i=1}^{n} \mathbf{AGAF} f_i) \rightarrow \psi.$$

$$M_{\{f_1,\ldots,f_n\}} \models \psi \not\Longleftrightarrow M \models (\bigwedge_{i=1}^{n} \mathbf{EGEF} f_i) \rightarrow \psi.$$

$\Longrightarrow$ We need specific procedures for Fair CTL Model Checking.

Remark: fair LTL M.C.

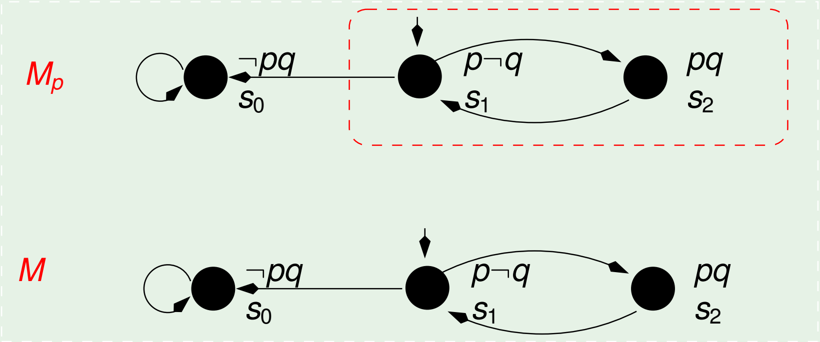
When model checking an LTL formula ψ, fairness conditions can be encoded into the formula:

$$M_{\{f_1,\ldots,f_n\}} \models \psi \Longleftrightarrow M \models (\bigwedge_{i=1}^{n} \mathbf{GF} f_i) \rightarrow \psi.$$

$\Longrightarrow$ There is no need for Fair LTL Model Checking procedures.

Ex. CTL: $M_{\{f_1,\ldots,f_n\}} \models \psi \not\Longleftrightarrow M \models (\bigwedge_{i=1}^{n} \mathbf{AGAF} f_i) \rightarrow \psi$.

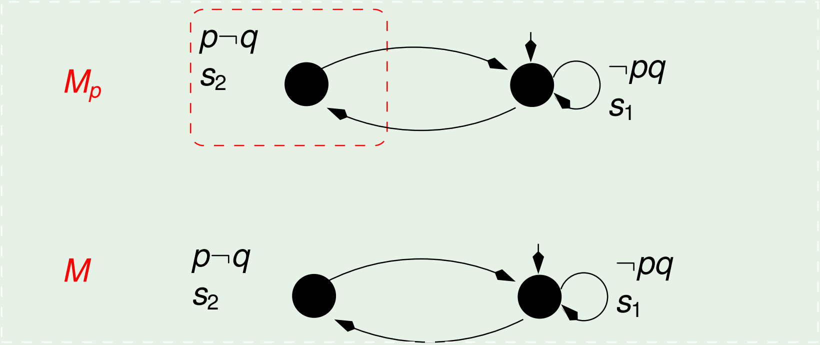
[Example provided by the student Davide Kirchner, 2014]



- $M_p \not\models \mathbf{AG}q$
- $M \models (\mathbf{AGAF}p) \rightarrow \mathbf{AG}q$

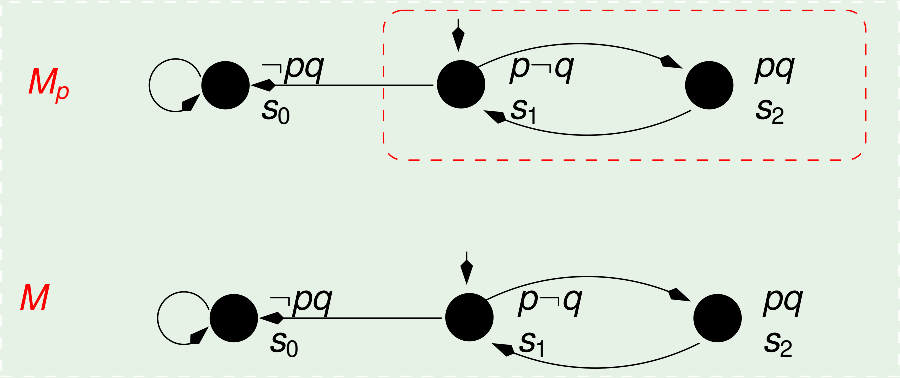
Ex. CTL: $M_{\{f_1,\dots,f_n\}} \models \psi \;\not\Longleftrightarrow\; M \models (\bigwedge_{i=1}^{n} \mathbf{EGEF} f_i) \rightarrow \psi$.

[Example provided by the student Daniele Giuliani, 2019]



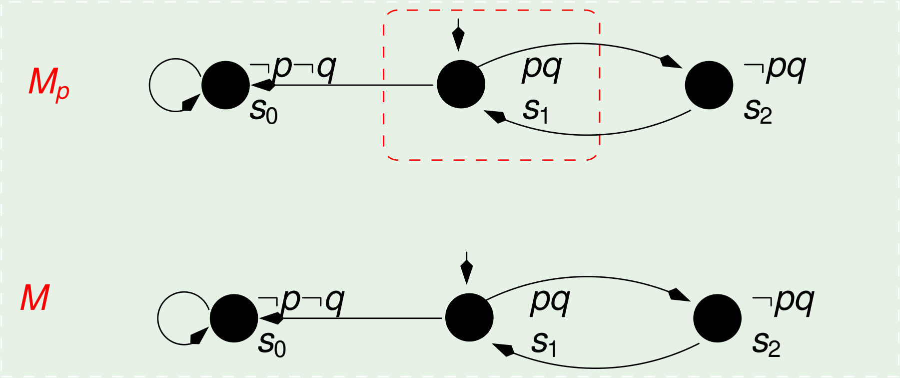
- $M_p \not\models \mathbf{EFEG}q$
- $M \models (\mathbf{EGEF}p) \rightarrow \mathbf{EFEG}q$

Ex. LTL (1): $M_{\{f_1,\dots,f_n\}} \models \psi \iff M \models (\bigwedge_{i=1}^{n} \mathbf{GF} f_i) \to \psi$.



- $M_p \not\models \mathbf{G}q$
- $M \not\models (\mathbf{GF}p) \to \mathbf{G}q$

Ex. LTL (2): $M_{\{f_1,\ldots,f_n\}} \models \psi \iff M \models (\bigwedge_{i=1}^{n} \mathbf{GF} f_i) \rightarrow \psi$.



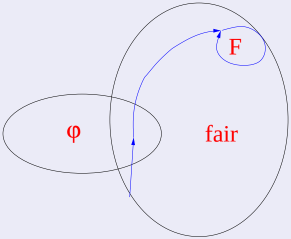
- $M_p \models \mathbf{G} q$
- $M \models (\mathbf{GF} p) \rightarrow \mathbf{G} q$

Fair CTL Model Checking

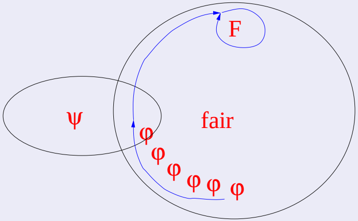
- In order to solve the fair CTL model checking problem, we must be able to compute:
  - $[\varphi_f]$ s.t. φ Boolean (i.e. $[\varphi]$ under fairness conditions f)
  - $[\mathbf{E}_f\mathbf{X}(\varphi)]$ (i.e. $[\mathbf{EX}\varphi]$ under fairness conditions f)
  - $[\mathbf{E}_f(\varphi\mathbf{U}\psi)]$ (i.e. $[\mathbf{E}(\varphi\mathbf{U}\psi)]$ under fairness conditions f)
  - $[\mathbf{E}_f\mathbf{G}\varphi]$ (i.e. $[\mathbf{EG}\varphi]$ under fairness conditions f).
- Suppose we have a procedure `Check_FairEG` to compute $[\mathbf{E}_f\mathbf{G}\varphi]$.
- Let $fair \stackrel{\text{def}}{=} \mathbf{E}_f\mathbf{G}true$. ($M, s \models \mathbf{E}_f\mathbf{G}true$ if s is a fair state.)
- if φ is Boolean, then $M_f, s \models \varphi$ iff $M, s \models (\varphi \wedge fair)$
- We can rewrite all the other fair operators:
  - $\mathbf{E}_f\mathbf{X}(\varphi) \equiv \mathbf{EX}(\varphi \wedge fair)$
  - $\mathbf{E}_f(\varphi\mathbf{U}\psi) \equiv \mathbf{E}(\varphi\mathbf{U}(\psi \wedge fair))$

Fair CTL Model Checking

- $\mathbf{E}_f\mathbf{X}(\varphi) \equiv \mathbf{EX}(\varphi \wedge fair)$:



- $\mathbf{E}_f(\varphi \mathbf{U} \psi) \equiv \mathbf{E}(\varphi \mathbf{U}(\psi \wedge fair))$:



Language-Emptiness Checking for Fair Kripke Models

Fair_CheckEG

Given: a fair Kripke model $M_F := \langle S, R, I, AP, L, F \rangle$ and a CTL formula φ s.t. $[\varphi] \subseteq S$, `Fair_CheckEG(`φ`)` returns the subset of the states s in $[\varphi]$ from which at least one fair path π entirely included in $[\varphi]$ passes through

`Fair_CheckEG(`*true*`)` computes the set of fair states of M_f

$\Longrightarrow$ $I \subseteq$ `Fair_CheckEG(`*true*`)` iff $\mathcal{L}(M_f) \neq \emptyset$

Ingredients (from CTL Model Checking)

Some primitive functions from CTL Model Checking:

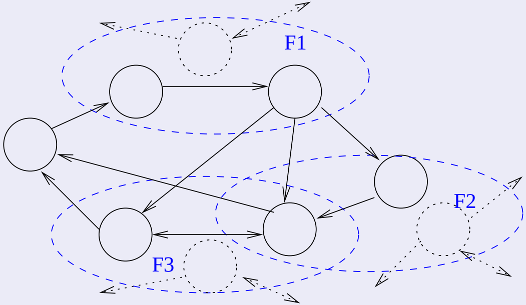
- `Check_EX`(ϕ): returns the set of states from which a path verifying $\mathbf{X}\phi$ holds (i.e., the preimage of the set of states where ϕ holds)
- `Check_EG`(ϕ): returns the set of states from which a path verifying $\mathbf{G}\phi$ holds
- `Check_EU`(ϕ_1, ϕ_2): returns the set of states from which a path verifying $\phi_1 \mathbf{U} \phi_2$ holds

Outline

SCC-based Check_FairEG

A Strongly Connected Component (SCC) of a directed graph is a maximal subgraph s.t. all its nodes are reachable from each other.

Given a fair Kripke model M, a fair non-trivial SCC is an SCC with at least one edge that contains at least one state for every fair condition
$\Longrightarrow$ all states in a fair (non-trivial) SCC are fair states



SCC-based Check_FairEG (cont.)

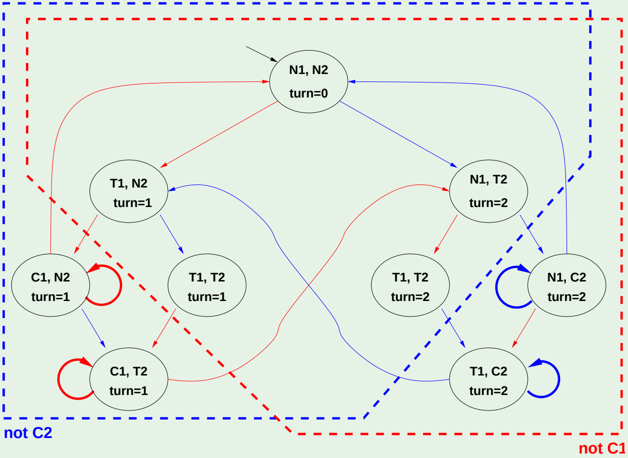
`Check_FairEG(`$[\phi]$`):`

(i) restrict the graph of M to $[\phi]$;
(ii) find all fair non-trivial SCCs C_i
(iii) build $C := \cup_i C_i$;
(iv) compute the states that can reach C (`Check_EU(`$[\phi]$`, `C`)`).

$[\phi]$: set of states where ϕ holds (aka denotation of ϕ)

Example: Check_FairEG

F := {{ not C1},{not C2}}



EG$\neg C_1$

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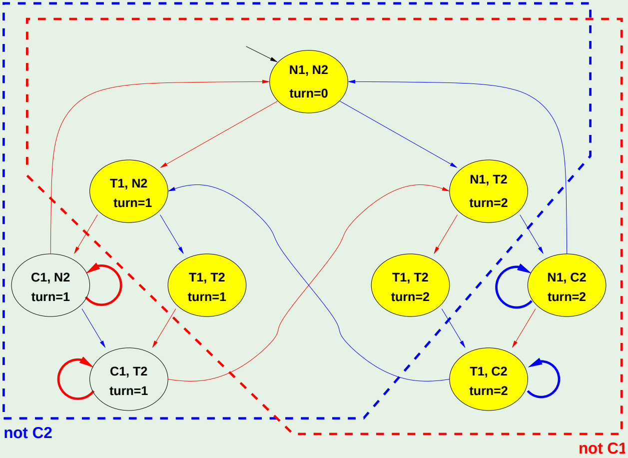
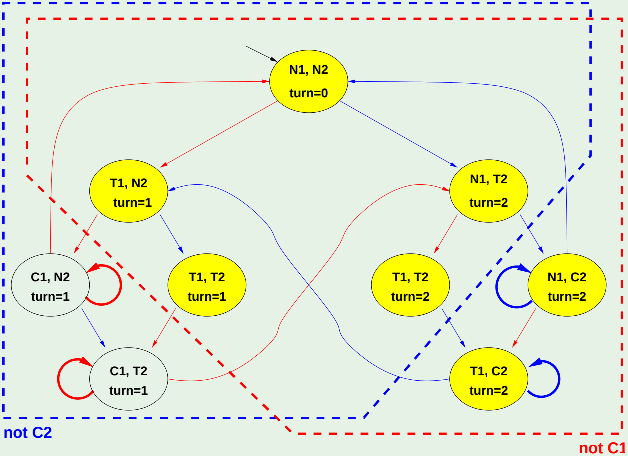


EG$\neg C_1$

Check_FairEG($\neg C_1$): 1. compute $[\neg C_1]$

Example: Check_FairEG

F := {{ not C1},{not C2}}

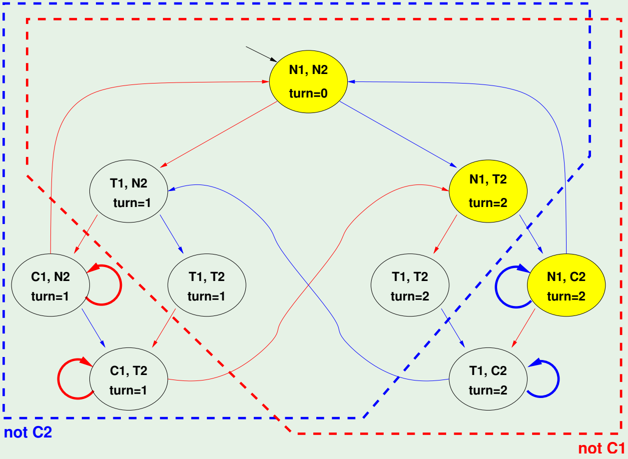


EG$\neg C_1$

Check_FairEG($\neg C_1$): 2. restrict the graph to $[\neg C_1]$

Example: Check_FairEG

F := {{ not C1},{not C2}}

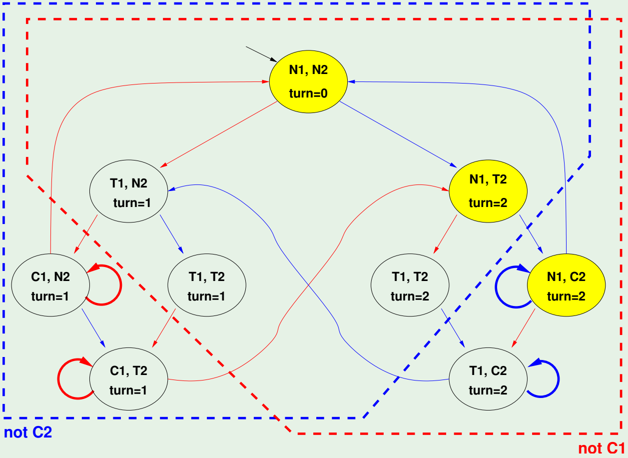


EG $\neg C_1$

Check_FairEG($\neg C_1$): 3. find all fair non-trivial SCC's

Example: Check_FairEG

F := {{ not C1},{not C2}}

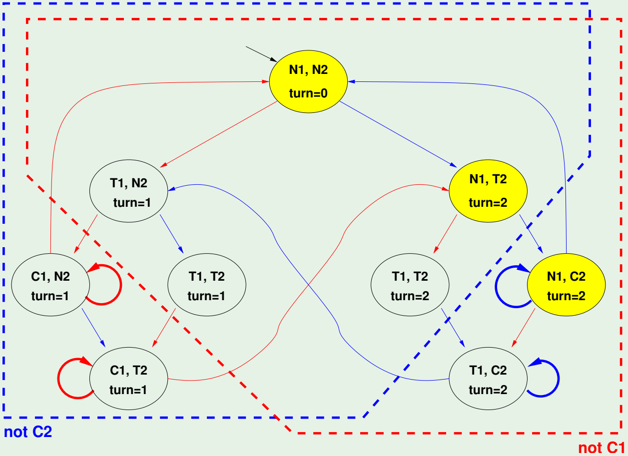


EG$\neg C_1$

Check_FairEG($\neg C_1$): 4. build the union C of all SCC's

Example: Check_FairEG

F := {{ not C1},{not C2}}



EG$\neg C_1$

Check_FairEG($\neg C_1$): 5. compute the states which can reach it

SCC-based Check_FairEG - Drawbacks

- SCCs computation requires a linear ($O(\#nodes + \#edges)$) DFS (Tarjan).
- The DFS manipulates the states explicitly, storing information for every state.
- A DFS is not suitable for symbolic model checking where we manipulate sets of states.

$\Longrightarrow$ We want an algorithm based on (symbolic) preimage computation.

Outline

Emerson-Lei Algorithm

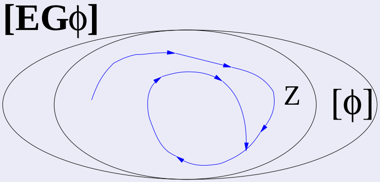
Fixpoint characterization of **EG** and fair **EG**

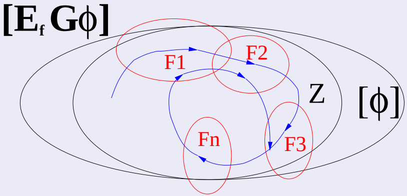
"$[\phi]$" denotes the set of states where ϕ holds

- Theorem (Emerson & Clarke): $[\mathbf{EG}\phi] = \nu Z.([\phi] \cap [\mathbf{EX}Z])$
  The greatest set Z s.t. every state z in Z satisfies ϕ and reaches another state in Z in one step.

We can characterize fair **EG** (aka "$\mathbf{E}_f\mathbf{G}$") similarly:

- Theorem (Emerson & Lei): $[\mathbf{E}_f\mathbf{G}\phi] = \nu Z.([\phi] \cap \bigcap_{F_i \in FT} [\mathbf{EX}\ \mathbf{E}(Z\mathbf{U}(Z \cap F_i))])$
  The greatest set Z s.t. every state z in Z satisfies ϕ and, for every set $F_i \in FT$, z reaches a state in $F_i \cap Z$ by means of a non-trivial path that lies in Z.





Emerson-Lei Algorithm

Recall: $[\mathbf{E}_f\mathbf{G}\phi] = \nu Z.([\phi] \cap \bigcap_{F_i \in FT}[\mathbf{EX}\ \mathbf{E}(Z\mathbf{U}(Z \cap F_i))])$

```
state_set Check_FairEG( state_set [φ]) {
    Z′ := [φ];
    repeat
        Z := Z′;
        for each F_i in FT
            Y := Check_EU(Z, F_i ∩ Z);
            Z′ := Z′ ∩ PreImage(Y));
        end for;
    until (Z′ = Z);
    return Z;
}
```

Implementation of the above formula

Emerson-Lei Algorithm

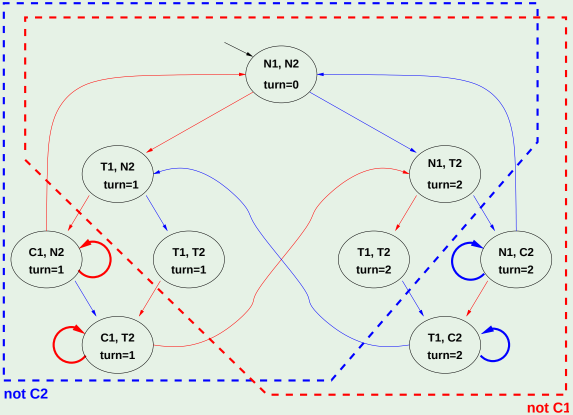
Recall: $[\mathbf{E}_f\mathbf{G}\phi] = \nu Z.([\phi] \cap \bigcap_{F_i \in FT}[\mathbf{EX}\ \mathbf{E}(Z\mathbf{U}(Z \cap F_i))])$

```
state_set Check_FairEG( state_set [φ]) {
    Z′ := [φ];
    repeat
        Z:= Z′;
        for each F_i in FT
            Y:= Check_EU(Z′, F_i ∩ Z′);
            Z′ := Z′ ∩ PreImage(Y));
        end for;
    until (Z′ = Z);
    return Z;
}
```

Slight improvement: do not consider states in $Z \setminus Z'$

Example: Check_FairEG

F := { { not C1},{not C2}}

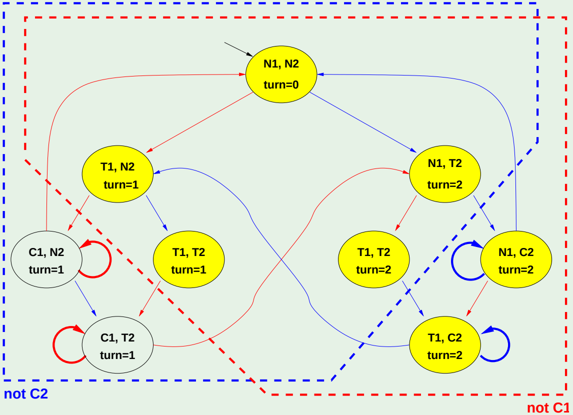


$[\mathbf{E}_f\mathbf{G}\neg C_1]$

Fixpoint reached

Example: Check_FairEG

F := { { not C1},{not C2}}

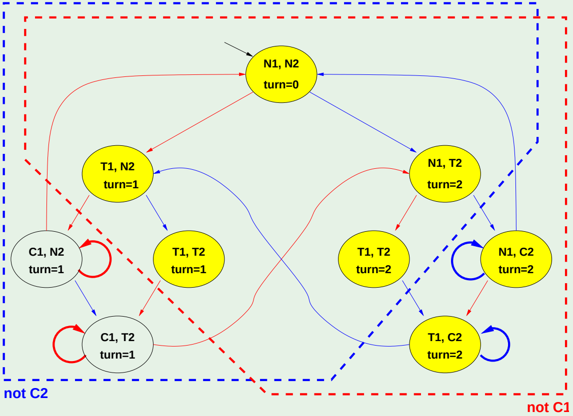


$[\mathbf{E}_f\mathbf{G} \neg C_1]$

Fixpoint reached

Example: Check_FairEG

F := { { not C1},{not C2}}



$[\mathbf{E}_f\mathbf{G}\neg C_1]$

$[\mathbf{E}_f\mathbf{G}g] = \nu Z.[g] \cap [\mathbf{EXE}(Z\mathbf{U}(Z \cap F_1))] \cap [\mathbf{EXE}(Z\mathbf{U}(Z \cap F_2))]$

Fixpoint reached

Example: Check_FairEG

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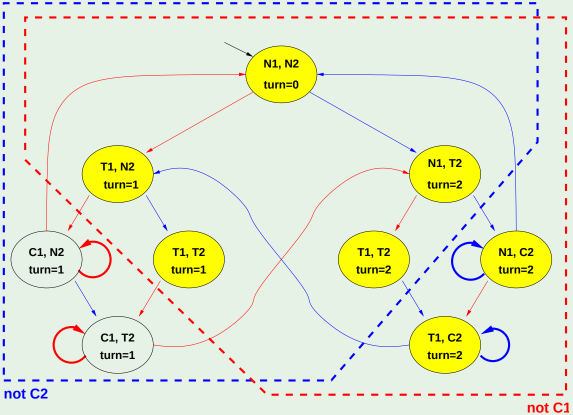

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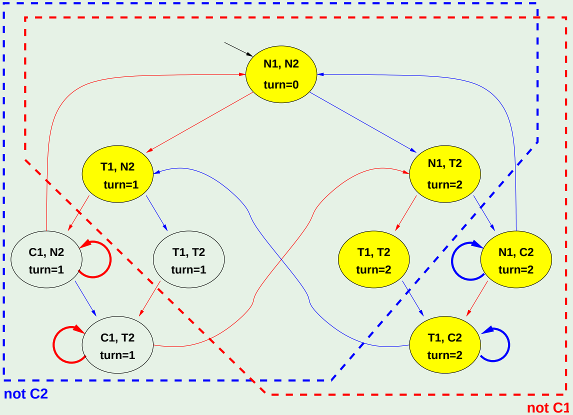

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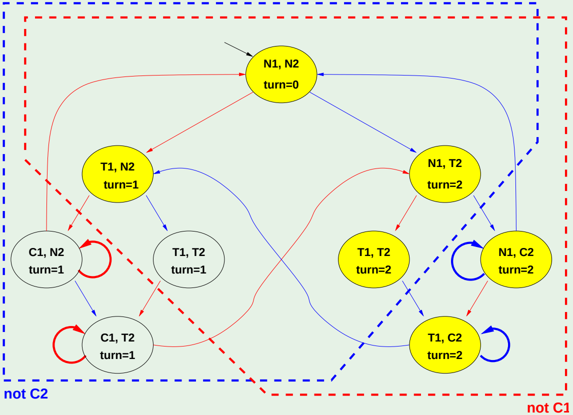

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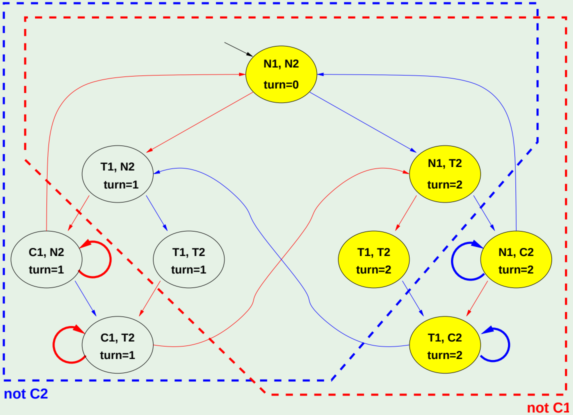

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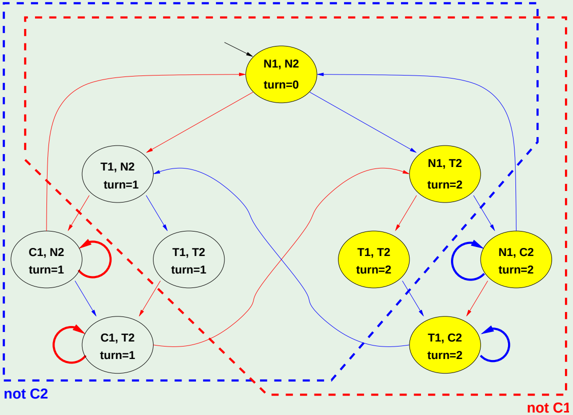

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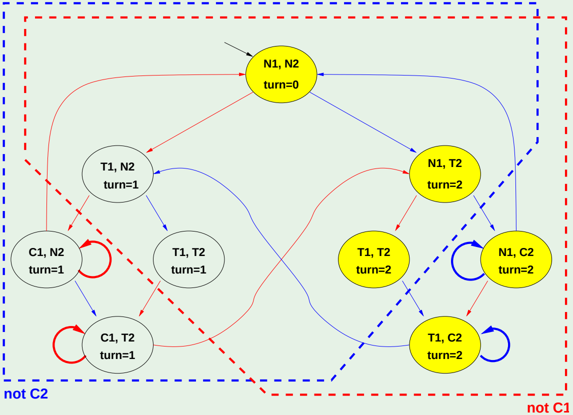

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$[\mathbf{E}_f\mathbf{G}g] = \nu Z.[g] \cap [\mathbf{EX}\mathbf{E}(Z\mathbf{U}(Z \cap F_1))] \cap [\mathbf{EX}\mathbf{E}(Z\mathbf{U}(Z \cap F_2))]$

Fixpoint reached

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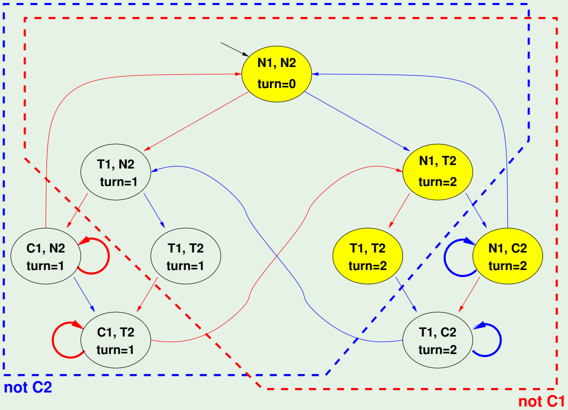

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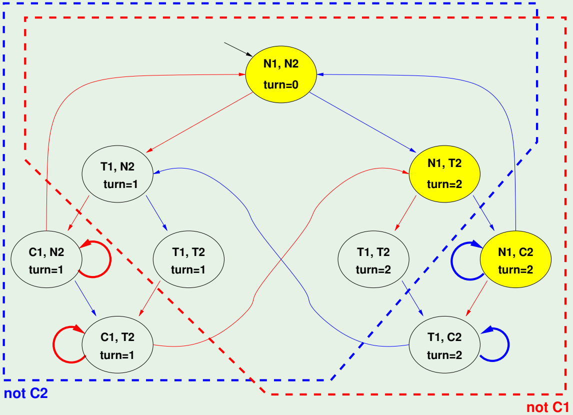

$[\mathbf{E}_f\mathbf{G}\neg C_1]$

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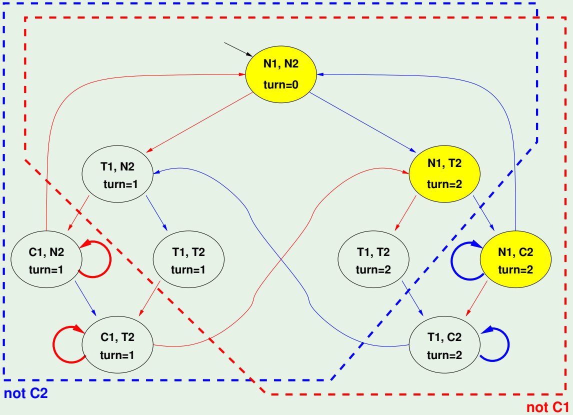


$[\mathbf{E}_f\mathbf{G}\neg C_1]$

$[\mathbf{E}_f\mathbf{G}g] = \nu Z.[g] \cap [\mathbf{EX}\mathbf{E}(Z\mathbf{U}(Z \cap F_1))] \cap [\mathbf{EX}\mathbf{E}(Z\mathbf{U}(Z \cap F_2))]$

Fixpoint reached

Outline

Outline

The Main Problem of M.C.: State Space Explosion

- The bottleneck:
  - Exhaustive analysis may require to store all the states of the Kripke structure, and to explore them one-by-one
  - The state space may be exponential in the number of components and variables
    - E.g., 300 Boolean vars $\Longrightarrow$ up to $2^{300} \approx 10^{100}$ states!
  - State Space Explosion:
    - too much memory required
    - too much CPU time required to explore each state
- A solution: Symbolic Model Checking

Symbolic Model Checking

Symbolic representation:

- manipulation of sets of states (rather than single states);
- sets of states represented by formulae in propositional logic;
  - set cardinality not directly correlated to size
- expansion of sets of transitions (rather than single transitions);

Symbolic Model Checking [cont.]

- Two main symbolic techniques:
  - Ordered Binary Decision Diagrams (OBDDs)
  - Propositional Satisfiability Checkers (SAT solvers)
- Different model checking algorithms:
  - Fix-point Model Checking (historically, for CTL)
  - Fix-point Model Checking for LTL (conversion to fair CTL MC)
  - Bounded Model Checking (historically, for LTL)
  - Invariant Checking
  - ...

Symbolic Representation of Kripke Models

- Symbolic representation:
  - sets of states as their characteristic function (Boolean formula)
  - provide logical representation and transformations of characteristic functions
- Example:
  - three state variables x_1, x_2, x_3:
    $\{$ 000, 001, 010, 011 $\}$ represented as “first bit false”: $\neg x_1$
  - with five state variables x_1, x_2, x_3, x_4, x_5:
    $\{$ 00000, 00001, 00010, 00011, 00100, 00101, 00110, 00111,. . . , 01111 $\}$ still represented as “first bit false”: $\neg x_1$

Kripke Models in Propositional Logic

- Let $M = (S, I, R, L, AF)$ be a Kripke model
- States $s \in S$ are described by means of an array V of Boolean state variables.
- A state is a truth assignment to each atomic proposition in V.
  - 0100 is represented by the formula $(\neg x_1 \wedge x_2 \wedge \neg x_3 \wedge \neg x_4)$
  - we call $\xi(s)$ the formula representing the state $s \in S$
    (Intuition: $\xi(s)$ holds iff the system is in the state s)
- A set of states $Q \subseteq S$ can be represented by any formula which is logically equivalent to the formula $\xi(Q)$:

$$\bigvee_{s \in Q} \xi(s)$$

  (Intuition: $\xi(Q)$ holds iff the system is in one of the states $s \in Q$)
- Bijection between models of $\xi(Q)$ and states in Q

Remark

- Every propositional formula is a (typically very compact) representation of the set of assignments satisfying it
- Any formula equivalent to $\xi(Q)$ is a representation of Q
  $\Longrightarrow$ Typically Q can be encoded by much smaller formulas than $\bigvee_{s \in Q} \xi(s)$!
- Example: $Q =$ { 00000, 00001, 00010, 00011, 00100, 00101, 00110, 00111, ..., 01111 } represented as "first bit false": $\neg x_1$

$$\bigvee_{s \in Q} \xi(s) \quad = \quad \left.\begin{array}{l} (\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5) \vee \\ (\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge x_5) \vee \\ (\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5) \vee \\ \ldots \\ (\neg x_1 \wedge x_2 \wedge x_3 \wedge x_4 \wedge x_5) \end{array}\right\} 2^4 \textit{disjuncts}$$

Symbolic Representation of Set Operators

One-to-one correspondence between sets and Boolean operators

- Set of all the states: $\xi(S) := \top$
- Empty set : $\xi(\emptyset) := \bot$
- Union represented by disjunction: $\xi(P \cup Q) := \xi(P) \vee \xi(Q)$
- Intersection represented by conjunction: $\xi(P \cap Q) := \xi(P) \wedge \xi(Q)$
- Complement represented by negation: $\xi(S/P) := \neg\xi(P)$

Symbolic Representation of Transition Relations

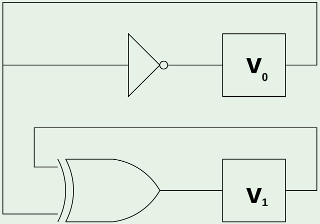
- The transition relation R is a set of pairs of states: $R \subseteq S \times S$
- A transition is a pair of states (s, s')
- A new vector of variables V' (the next state vector) represents the value of variables after the transition has occurred
- $\xi(s, s')$ defined as $\xi(s) \wedge \xi(s')$ (Intuition: $\xi(s, s')$ holds iff the system is in the state s and moves to state s' in next step)
- The transition relation R can be represented by any formula equivalent to:

$$\bigvee_{(s,s')\in R} \xi(s, s') = \bigvee_{(s,s')\in R} (\xi(s) \wedge \xi(s'))$$

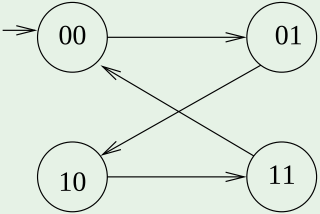
Each formula equivalent to $\xi(R)$ is a representation of R
$\Longrightarrow$ Typically R can be encoded by a much smaller formula than $\bigvee_{(s,s')\in R} \xi(s) \wedge \xi(s')$!

Example: a simple counter

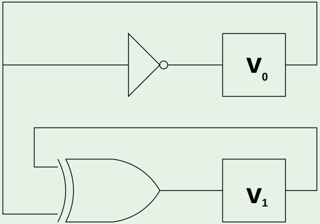
```
MODULE main
 VAR
   v0     : boolean;
   v1     : boolean;
   out    : 0..3;

 ASSIGN
   init(v0)  := 0;
   next(v0)  := !v0;

   init(v1)  := 0;
   next(v1)  := (v0 xor v1);

   out := toint(v0) + 2*toint(v1);
```



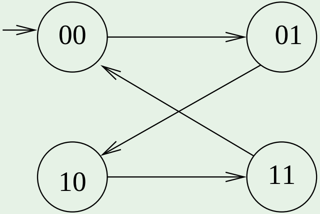
v_1	v_0	v_1'	v_0'
0	0	0	1
0	1	1	0
1	0	1	1
1	1	0	0



Example: a simple counter [cont.]



v_1	v_0	v_1'	v_0'
0	0	0	1
0	1	1	0
1	0	1	1
1	1	0	0

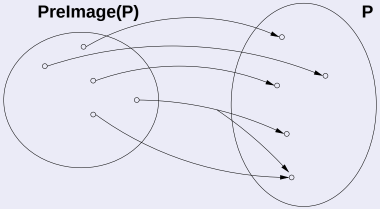


$$\xi(R) \quad = \quad (v_0' \leftrightarrow \neg v_0) \land (v_1' \leftrightarrow v_0 \bigoplus v_1)$$

$$\bigvee_{(s,s')\in R} \xi(s) \land \xi(s') \quad = \quad \begin{array}{l} (\neg v_1 \land \neg v_0 \land \neg v_1' \land v_0') \ \lor \\ (\neg v_1 \land v_0 \land v_1' \land \neg v_0') \ \lor \\ (v_1 \land \neg v_0 \land v_1' \land v_0') \ \lor \\ (v_1 \land v_0 \land \neg v_1' \land \neg v_0') \end{array}$$

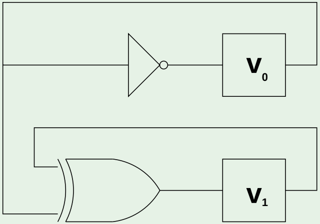
Pre-Image

- (Backward) pre-image of a set of states:

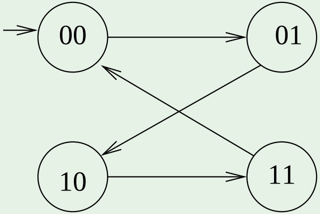


  Evaluate one-shot all transitions ending in the states of the set
- Set theoretic view: $PreImage(P, R) := \{s \mid \text{for some } s' \in P, (s, s') \in R\}$
- Logical view: $\xi(PreImage(P, R)) \ := \ \exists V'.(\xi(P)[V'] \wedge \xi(R)[V, V'])$
- μ over V is s.t $\mu \models \exists V'.(\xi(P)[V'] \wedge \xi(R)[V, V'])$ iff,
  for some μ' over V', we have: $\mu \cup \mu' \models (\xi(P)[V'] \wedge \xi(R)[V, V'])$,
  i.e., $\mu' \models \xi(P)[V']$ and $\mu \cup \mu' \models \xi(R)[V, V'])$
  - Intuition: $\mu \Longleftrightarrow s$, $\mu' \Longleftrightarrow s'$, $\mu \cup \mu' \Longleftrightarrow \langle s, s' \rangle$

Example: simple counter



v_1	v_0	v_1'	v_0'
0	0	0	1
0	1	1	0
1	0	1	1
1	1	0	0

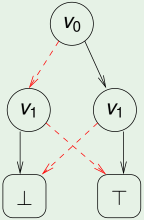


$\xi(R) = (v_0' \leftrightarrow \neg v_0) \wedge (v_1' \leftrightarrow v_0 \bigoplus v_1)$

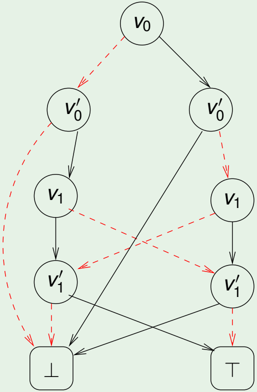
$\xi(P) := (v_0 \leftrightarrow v_1)$ (i.e., $P = \{00, 11\}$)

$$\begin{aligned}
&\xi(PreImage(P, R)) &= \\
&\exists V'.(\xi(P)[V'] \wedge \xi(R)[V, V']) &= \\
&\exists v_0' v_1'.((v_0' \leftrightarrow v_1') \wedge (v_0' \leftrightarrow \neg v_0) \wedge (v_1' \leftrightarrow v_0 \bigoplus v_1)) &= \\
&\underbrace{(\neg v_0 \wedge v_0 \bigoplus v_1)}_{v_0'=\top, v_1'=\top} \vee \underbrace{\bot}_{v_0'=\top, v_1'=\bot} \vee \underbrace{\bot}_{v_0'=\bot, v_1'=\top} \vee \underbrace{(v_0 \wedge \neg(v_0 \bigoplus v_1))}_{v_0'=\bot, v_1'=\bot} &= \\
&v_1 \quad (i.e., \{10, 11\})
\end{aligned}$$

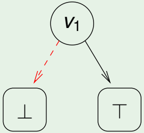
Pre-Image [cont.]



$\xi(P) = v_0 \leftrightarrow v_1$



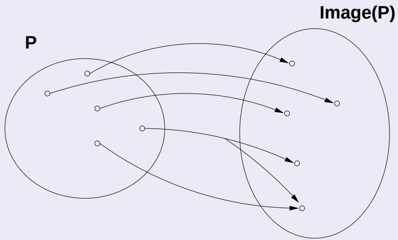
$\xi(R) = (v_0' \leftrightarrow \neg v_0) \wedge (v_1' \leftrightarrow v_0 \bigoplus v_1)$



$$\xi(PreImage(P, R)) = \\ \exists V'.((v_0' \leftrightarrow v_1') \wedge (v_0' \leftrightarrow \neg v_0) \wedge (v_1' \leftrightarrow v_0 \bigoplus v_1)) = \\ v_1$$

Forward Image

- Forward image of a set:



Evaluate one-shot all transitions from the states of the set

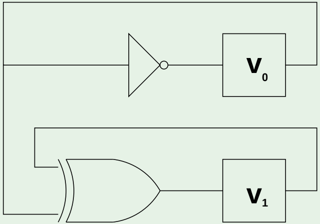
- Set theoretic view:

$$Image(P, R) := \{s' \mid \text{for some } s \in P, (s, s') \in R\}$$

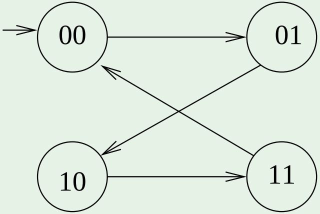
- Logical Characterization:

$$\xi(Image(P, R)) \;:=\; \exists V.(\xi(P)[V] \wedge \xi(R)[V, V'])$$

Example: simple counter



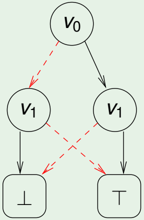
v_1	v_0	v_1'	v_0'
0	0	0	1
0	1	1	0
1	0	1	1
1	1	0	0



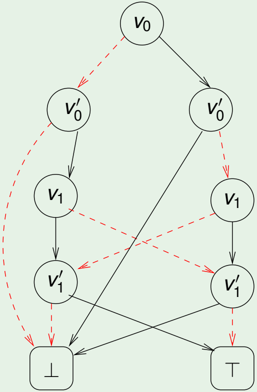
$\xi(R) = (v_0' \leftrightarrow \neg v_0) \wedge (v_1' \leftrightarrow v_0 \bigoplus v_1)$
$\xi(P) := (v_0 \leftrightarrow v_1)$ (i.e., $P = \{00, 11\}$)

$$\begin{aligned}
\xi(Image(P,R)) &= \exists V.(\xi(P)[V] \wedge \xi(R)[V, V']) \\
&= \exists V.((v_0 \leftrightarrow v_1) \wedge (v_0' \leftrightarrow \neg v_0) \wedge (v_1' \leftrightarrow v_0 \bigoplus v_1)) \\
&= \ldots \\
&= \neg v_1' \quad (i.e., \{00, 01\})
\end{aligned}$$

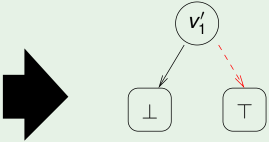
Forward Image [cont.]



$\xi(P) = v_0 \leftrightarrow v_1$



$\xi(R) = (v_0' \leftrightarrow \neg v_0) \wedge (v_1' \leftrightarrow v_0 \bigoplus v_1)$



$$\xi(Image(P, R)) = \\ \exists V.((v_0 \leftrightarrow v_1) \wedge (v_0' \leftrightarrow \neg v_0) \wedge (v_1' \leftrightarrow v_0 \bigoplus v_1)) = \\ \neg v_1'$$

Application of the Transition Relation

- Image and PreImage of a set of states S computed by means of quantified Boolean formulae
- The whole set of transitions can be fired (either forward or backward) in one logical operation
- The symbolic computation of PreImage and Image provide the primitives for symbolic search of the state space of FSM's

Notation Remark

Henceforth, for readability sake, we omit the "$\xi()$" notation in symbolic representations of systems.

- Kripke models represented as $\langle I(V), R(V, V')\rangle$
- Fair Kripke models represented as $\langle I(V), R(V, V'), F(V)\rangle$ s.t. $F(V) \stackrel{\text{def}}{=} \{F_1(V), .., F_k(V)\}$

Outline

```
STATE-SET Check(CTL_formula β) {
    case β of
    ⊤:           return S;
    ⊥:           return ∅;
    ¬β1:         return S\Check(β1);
    β1 ∧ β2:     return (Check(β1) ∩ Check(β2));
    EXβ1:        return PreImage(Check(β1));
    EGβ1:        return Check_EG(Check(β1));
    E(β1Uβ2):    return Check_EU(Check(β1),Check(β2));
}
```

General Symbolic CTL MC Procedure

```
OBDD        Check(CTL_formula β) {
    if (In_OBDD_Hash(β)) return OBDD_Get_From_Hash(β);
    case β of
    ⊤:            return obdd_true;
    ⊥:            return obdd_false;
    ¬β1:          return ¬ Check(β1);
    β1 ∧ β2:      return (Check(β1) ∧ Check(β2));
    EXβ1:         return PreImage(Check(β1));
    EGβ1:         return Check_EG(Check(β1));
    E(β1Uβ2):     return Check_EU(Check(β1),Check(β2));
}
```

Ingredients

Some primitive functions from CTL Model Checking:

- Symbolic `Check_EX`(ϕ):
  returns an OBDD representing the set of states from which a path verifying $\mathbf{X}\phi$ begins
  (i.e., the symbolic preimage of the set of states where ϕ holds)
- Symbolic `Check_EG`(ϕ):
  returns an OBDD representing the set of states from which a path verifying $\mathbf{G}\phi$ begins
- Symbolic `Check_EU`(ϕ_1, ϕ_2):
  returns an OBDD representing the set of states from which a path verifying $\phi_1 \mathbf{U} \phi_2$ begins

Check_EX

Explicit-state

State Set Check_EX(**State Set** X)
  return $\{s \mid \text{for some } s' \in X, (s, s') \in R\}$;

Symbolic

OBDD Check_EX(**OBDD** X)
  return $\exists V'.(\ X[V'] \wedge R[V, V'])$;

Same as Pre-Image computation.

Check_EG

Explicit-State

State Set Check_EG(**State Set** X)
    $Y' := X$;
    repeat
        $Y := Y'$;
        $Y' := Y \cap Check\_EX(Y)$;
    until $(Y' = Y)$;
return Y;

Symbolic

OBDD Check_EG(**OBDD** X)
    $Y' := X$;
    repeat
        $Y := Y'$;
        $Y' := Y \wedge Check\_EX(Y)$;
    until $(Y' \leftrightarrow Y)$;
return Y;

Hint (tableaux rule): $s \models$ **EG**ϕ only if $s \models \phi \wedge$ **EXEG**ϕ

Check_EU

Explicit-State

State Set Check_EU(**State Set** X_1,X_2)
$\quad Y' := X_2;$
$\quad$ **repeat**
$\quad\quad Y := Y';$
$\quad\quad Y' := Y \cup (X_1 \cap \mathit{Check\_EX}(Y));$
$\quad$ **until** $(Y' = Y);$
return $Y;$

Symbolic

OBDD Check_EU(**OBDD** X_1,X_2)
$\quad Y' := X_2;$
$\quad$ **repeat**
$\quad\quad Y := Y';$
$\quad\quad Y' := Y \vee (X_1 \wedge \mathit{Check\_EX}(Y));$
$\quad$ **until** $(Y' \leftrightarrow Y);$
return $Y;$

Hint (tableaux rule): $s \models \mathbf{E}(\phi_1 \mathbf{U} \phi_2)$ if $s \models \phi_2 \vee (\phi_1 \wedge \mathbf{EXE}(\phi_1 \mathbf{U} \phi_2))$

Outline

Language-Emptiness Checking for Fair Kripke Models

Fair_CheckEG

Given: a fair Kripke model $M_F := \langle S, R, I, AP, L, F \rangle$ and a CTL formula φ s.t. $[\varphi] \subseteq S$,
`Fair_CheckEG`(φ) returns the subset of the states s in $[\varphi]$ from which at least one fair path π entirely included in $[\varphi]$ passes through

Symbolic Fair_CheckEG

Given: the symbolic representation of a fair Kripke model $M_F := \langle I, R, F \rangle$
and a Boolean formula (OBDD) Ψ,
`Fair_CheckEG`(Ψ) returns a Boolean formula (OBDD) representing the subset of the states s in Ψ from which at least one fair path π entirely included in Ψ passes through

`Fair_CheckEG`(*true*) computes (the symbolic representation of) the set of fair states of M_f
$\Longrightarrow$ $I \subseteq$ `Fair_CheckEG`(*true*) iff $\mathcal{L}(M_f) \neq \emptyset$

Ingredients (from Symbolic CTL Model Checking)

Some primitive functions from CTL Model Checking:

- Symbolic `Check_EX`(ϕ): returns an OBDD representing the set of states from which a path verifying $\mathbf{X}\phi$ begins

  (i.e., the symbolic preimage of the set of states where ϕ holds)
- Symbolic `Check_EG`(ϕ): returns an OBDD representing the set of states from which a path verifying $\mathbf{G}\phi$ begins
- Symbolic `Check_EU`(ϕ_1, ϕ_2): returns an OBDD representing the set of states from which a path verifying $\phi_1\mathbf{U}\phi_2$ begins

Emerson-Lei Algorithm

Recall: $[\mathbf{E}_f\mathbf{G}\phi] = \nu Z.([\phi] \cap \bigcap_{F_i \in FT}[\mathbf{EX}\ \mathbf{E}(Z\mathbf{U}(Z \cap F_i))])$

```
state_set Check_FairEG( state_set [φ]) {
   Z′ := [φ];
   repeat
      Z:= Z′;
      for each Fi in FT
         Y:= Check_EU(Z′,Fi ∩ Z′);
         Z′ := Z′ ∩ PreImage(Y));
      end for;
   until (Z′ = Z);
   return Z;
}
```

Slight improvement: do not consider states in $Z \setminus Z'$

Emerson-Lei Algorithm (symbolic version)

Recall: $[\mathbf{E}_f\mathbf{G}\phi] = \nu Z.([\phi] \cap \bigcap_{F_i \in FT}[\mathbf{EX}\ \mathbf{E}(Z\mathbf{U}(Z \wedge F_i))])$

Obdd `Check_FairEG(` **Obdd** ϕ`) {`
    `Z' :=` ϕ`;`
    repeat
        `Z:= Z';`
        for each F_i **in** `FT`
            `Y:= Check_EU(Z',`F_i $\wedge$ `Z');`
            `Z' := Z'` $\wedge$ `PreImage(Y));`
        end for;
    until `(Z'` $\leftrightarrow$ `Z);`
    return `Z;`
`}`

Symbolic version.

Outline

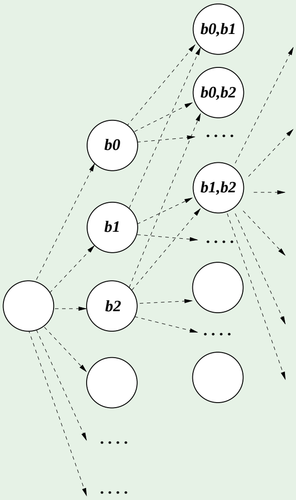
A simple example

```
MODULE main
VAR
  b0 : boolean;
  b1 : boolean;
  ...
ASSIGN
  init(b0) := 0;
  next(b0) := case
    b0  : 1;
    !b0 : {0,1};
  esac;
  init(b1) := 0;
  next(b1) := case
    b1  : 1;
    !b1 : {0,1};
  esac;
  ...
```

A simple example [cont.]

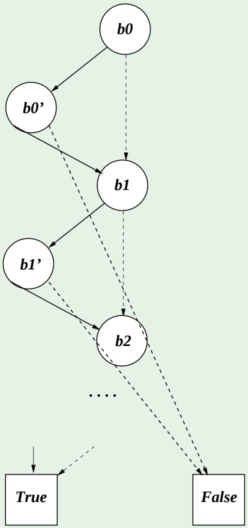
- N Boolean variables $b0, b1, \ldots$
- Initially, all variables set to 0
- Each variable can pass from 0 to 1, but not vice-versa
- 2^N states, all reachable
- (Simplified) model of a student career behaviour.

A simple example: FSM



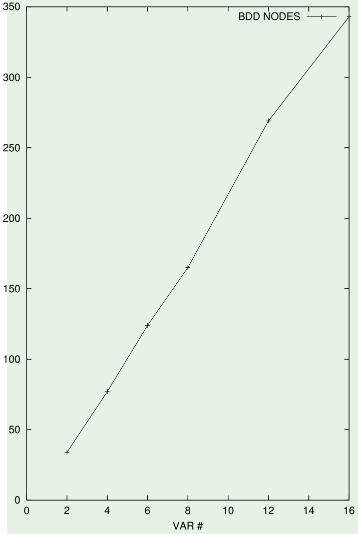
(transitive transitons omitted)
2^N STATES
$O(2^N)$ TRANSITIONS

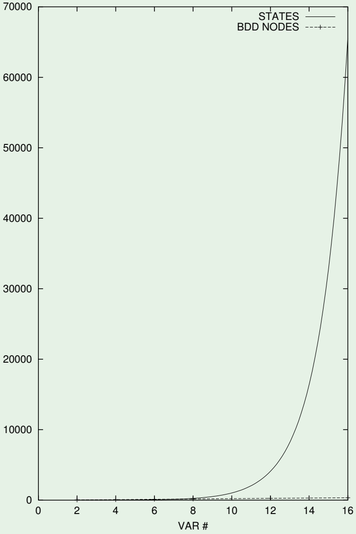
A simple example: $OBDD(\xi(R))$



$2N + 2$ NODES

A simple example: states vs. OBDD nodes [NuSMV.2]

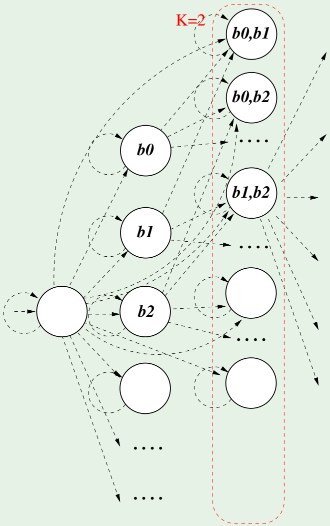




A simple example: reaching K bits true

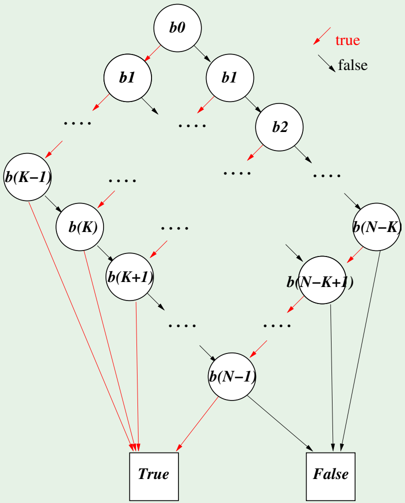
- Property $\mathbf{EF}(b0 + b1 + ... + b(N-1) \geq K)$ $(K \leq N)$
  (it may be reached a state in which K bits are true)
- E.g.: "it is reachable a state where K exams are passed"

A simple example: FSM



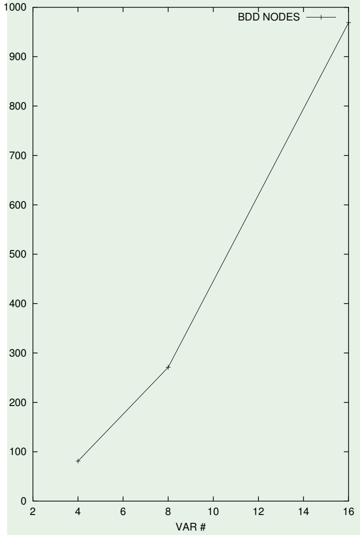
$$\binom{N}{K} + \binom{N}{K+1} + \ldots + \binom{N}{N}$$

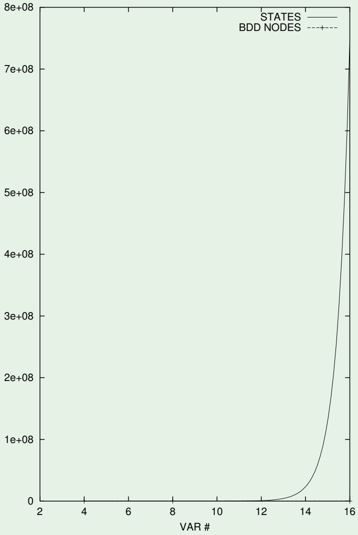
A simple example: $OBDD(\xi(\varphi))$



$(N - K + 1) \cdot K + 2$ NODES

A simple example: states vs. OBDD nodes [NuSMV.2]





Outline

Outline

Symbolic LTL Satisfiability and Entailment

LTL Validity/Satisfiability

- Let ψ be an LTL formula

$$
\begin{array}{rl}
 & \models \psi \quad \text{(LTL)} \\
\iff & \neg\psi \text{ unsat} \\
\iff & \mathcal{L}(T_{\neg\psi}) = \emptyset
\end{array}
$$

- $T_{\neg\psi}$ is a fair Kripke model (aka tableaux) which represents all and only the paths that satisfy $\neg\psi$ (do not satisfy ψ)

LTL Entailment

- Let φ, ψ be an LTL formula

$$
\begin{array}{rl}
 & \varphi \models \psi \quad \text{(LTL)} \\
 & \models \varphi \rightarrow \psi \quad \text{(LTL)} \\
\iff & \varphi \wedge \neg\psi \text{ unsat} \\
\iff & \mathcal{L}(T_{\varphi\wedge\neg\psi}) = \emptyset
\end{array}
$$

- $T_{\varphi\wedge\neg\psi}$ is a fair Kripke model (aka tableaux) which represents all and only the paths that satisfy $\varphi \wedge \neg\psi$ (satisfy φ and do not satisfy ψ)

Symbolic LTL Model Checking

LTL Model Checking

- Let M be a Kripke model and ψ be an LTL formula

$$
\begin{aligned}
& M \models \psi \quad \text{(LTL)} \\
\Longleftrightarrow \ & \mathcal{L}(M) \subseteq \mathcal{L}(\psi) \\
\Longleftrightarrow \ & \mathcal{L}(M) \cap \overline{\mathcal{L}(\psi)} = \emptyset \\
\Longleftrightarrow \ & \mathcal{L}(M) \cap \mathcal{L}(\neg\psi) = \emptyset \\
\Longleftrightarrow \ & \mathcal{L}(M) \cap \mathcal{L}(T_{\neg\psi}) = \emptyset \\
\Longleftrightarrow \ & \mathcal{L}(M \times T_{\neg\psi}) = \emptyset
\end{aligned}
$$

- $T_{\neg\psi}$ is a fair Kripke model (aka tableaux) which represents all and only the paths that satisfy $\neg\psi$ (do not satisfy ψ)

$\Longrightarrow$ $M \times T_{\neg\psi}$ represents all and only the paths appearing in M and not in ψ.

Symbolic LTL Model Checking

Three steps

Let $\varphi \stackrel{\text{def}}{=} \neg\psi$:

(i) Compute T_φ

(ii) Compute the product $M \times T_\varphi$

(iii) Check the emptiness of $\mathcal{L}(M \times T_\varphi)$

Outline

The Set of States

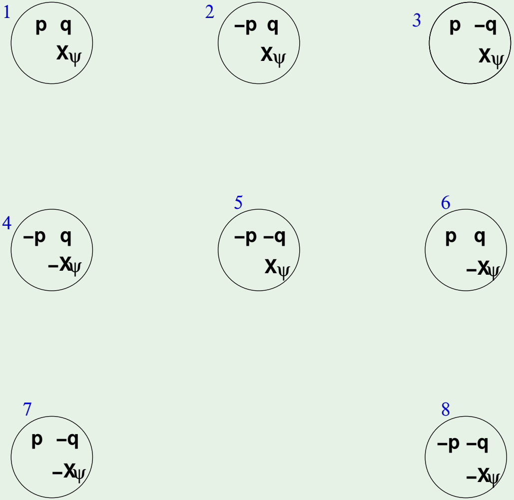
- Elementary subformulas of ψ: $el(\psi)$
  - $el(p) := \{p\}$
  - $el(\neg\varphi_1) := el(\varphi_1)$
  - $el(\varphi_1 \wedge \varphi_2) := el(\varphi_1) \cup el(\varphi_2)$
  - $el(\mathbf{X}\varphi_1) = \{\mathbf{X}\varphi_1\} \cup el(\varphi_1)$
  - $el(\varphi_1 \mathbf{U} \varphi_2) := \{\mathbf{X}(\varphi_1 \mathbf{U} \varphi_2)\} \cup el(\varphi_1) \cup el(\varphi_2)$
- Intuition: $el(\psi)$ is the set of propositions and **X**-formulas occurring ψ', ψ' being the result of applying recursively the tableau expansion rules to ψ
- The set of states S_{T_ψ} of T_ψ is given by $2^{el(\psi)}$
- The labeling function L_{T_ψ} of T_ψ comes straightforwardly (the label is the Boolean component of each state)

Example: $\psi := p\mathbf{U}q$

- $el(p\mathbf{U}q) = el((q \vee (p \wedge \mathbf{X}(p\mathbf{U}q))) = \{p, q, \mathbf{X}(p\mathbf{U}q)\}$

$$
\Longrightarrow S_{T_\psi} = \{
\begin{array}{lll}
1: & \{p, q, \mathbf{X}(p\mathbf{U}q)\}, & [p\mathbf{U}q] \\
2: & \{\neg p, q, \mathbf{X}(p\mathbf{U}q)\}, & [p\mathbf{U}q] \\
3: & \{p, \neg q, \mathbf{X}(p\mathbf{U}q)\}, & [p\mathbf{U}q] \\
4: & \{\neg p, q, \neg\mathbf{X}(p\mathbf{U}q)\}, & [p\mathbf{U}q] \\
5: & \{\neg p, \neg q, \mathbf{X}(p\mathbf{U}q)\}, & [\neg p\mathbf{U}q] \\
6: & \{p, q, \neg\mathbf{X}(p\mathbf{U}q)\}, & [p\mathbf{U}q] \\
7: & \{p, \neg q, \neg\mathbf{X}(p\mathbf{U}q)\}, & [\neg p\mathbf{U}q] \\
8: & \{\neg p, \neg q, \neg\mathbf{X}(p\mathbf{U}q)\} & [\neg p\mathbf{U}q]
\end{array}
\}
$$

Example: $\psi := p\mathbf{U}q$ [cont.]



$sat()$

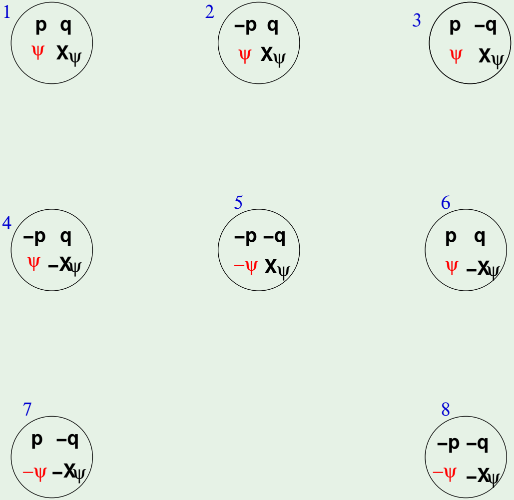
- Set of states in S_{T_ψ} satisfying φ_i: $sat(\varphi_i)$
  - $sat(\varphi_1) := \{s \mid \varphi_1 \in s\}$, $\varphi_1 \in el(\psi)$
  - $sat(\neg\varphi_1) := S_{T_\psi} / sat(\varphi_1)$
  - $sat(\varphi_1 \wedge \varphi_2) := sat(\varphi_1) \cap sat(\varphi_2)$
  - $sat(\varphi_1 \mathbf{U} \varphi_2) := sat(\varphi_2) \cup (sat(\varphi_1) \cap sat(\mathbf{X}(\varphi_1 \mathbf{U} \varphi_2)))$
- intuition: $sat()$ establishes in which states subformulas are true

Remark

- Semantics of "$\varphi_1 \mathbf{U} \varphi_2$" here induced by tableaux rule: $\varphi_1 \mathbf{U} \varphi_2 \stackrel{\text{def}}{=} \varphi_2 \vee (\varphi_1 \wedge \mathbf{X}(\varphi_1 \mathbf{U} \varphi_2))$

$\Longrightarrow$ weaker than standard semantics (aka "weak until", "$\varphi_1 \mathbf{W} \varphi_2$"):
a path where φ_1 is always true and φ_2 is always false satisfies it

Example: $\psi := p\mathbf{U}q$ [cont.]



Initial States and Transition Relation

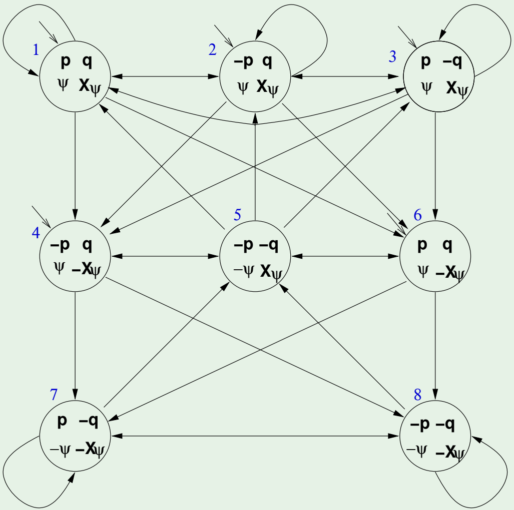
- Set of states in S_{T_ψ} satisfying φ_i: $sat(\varphi_i)$
  - $sat(\varphi_1) := \{s \mid \varphi_1 \in s\}$, $\varphi_1 \in el(\psi)$
  - $sat(\neg\varphi_1) := S_{T_\psi} / sat(\varphi_1)$
  - $sat(\varphi_1 \wedge \varphi_2) := sat(\varphi_1) \cap sat(\varphi_2)$
  - $sat(\varphi_1 \mathbf{U} \varphi_2) := sat(\varphi_2) \cup (sat(\varphi_1) \cap sat(\mathbf{X}(\varphi_1 \mathbf{U} \varphi_2)))$
- Intuition: $sat()$ establishes in which states subformulas are true
- The set of initial states I_{T_ψ} is defined as

$$I_{T_\psi} = sat(\psi)$$

- The transition relation R_{T_ψ} is defined as

$$R_{T_\psi}(s, s') = \bigcap_{\mathbf{X}\varphi_i \in el(\psi)} \{(s, s') \mid s \in sat(\mathbf{X}\varphi_i) \Leftrightarrow s' \in sat(\varphi_i)\}$$

Example: $\psi := p\mathbf{U}q$ [cont.]



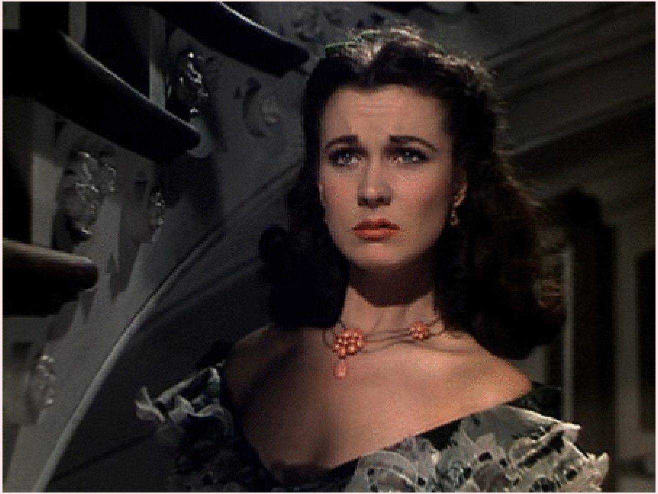
Problems with **U**-subformulas

- R_{T_ψ} does not guarantee that the **U**-subformulas are fulfilled
- Example: state 3 $\{p, \neg q, \mathbf{X}(p\mathbf{U}q)\}$:
  although state 3 belongs to

$$sat(p\mathbf{U}q) := sat(q) \cup (sat(p) \cap sat(\mathbf{X}(p\mathbf{U}q))),$$

the path which loops forever in state 3 does not satisfy $p\mathbf{U}q$, as q never holds in that path.

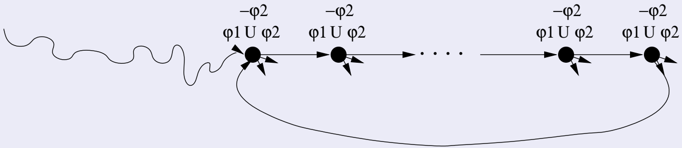
Tableaux Rules: a Quote



“After all... tomorrow is another day."
[Scarlett O’Hara, “Gone with the Wind”]

Fairness conditions for every **U**-subformula

- It must never happen that we get into
  a state s' from which we can enter a path π' in which $\varphi_1 \mathbf{U} \varphi_2$ holds forever and φ_2 never holds.

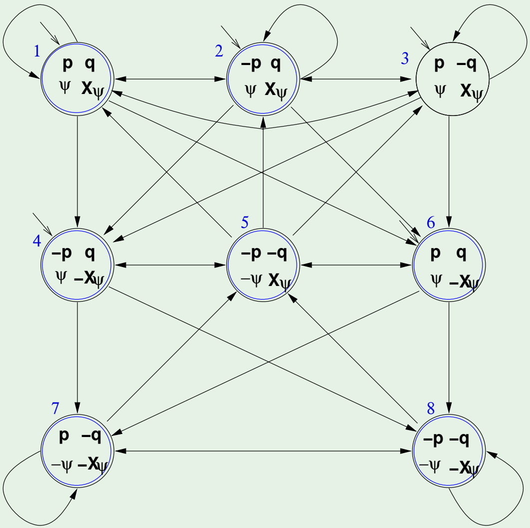


$\Longrightarrow$ For every [positive] **U**-subformula $\varphi_1 \mathbf{U} \varphi_2$ of ψ, we must add a fairness LTL condition $\mathbf{GF}(\neg(\varphi_1 \mathbf{U} \varphi_2) \vee \varphi_2)$
If no [positive] U-subformulas, then add one fairness condition $\mathbf{GF}\top$.

$\Longrightarrow$ We restrict the admissible paths of T_ψ to those which verify the fairness condition:
$T_\psi := \langle S_{T_\psi}, I_{T_\psi}, R_{T_\psi}, L_{T_\psi}, F_{T_\psi} \rangle$

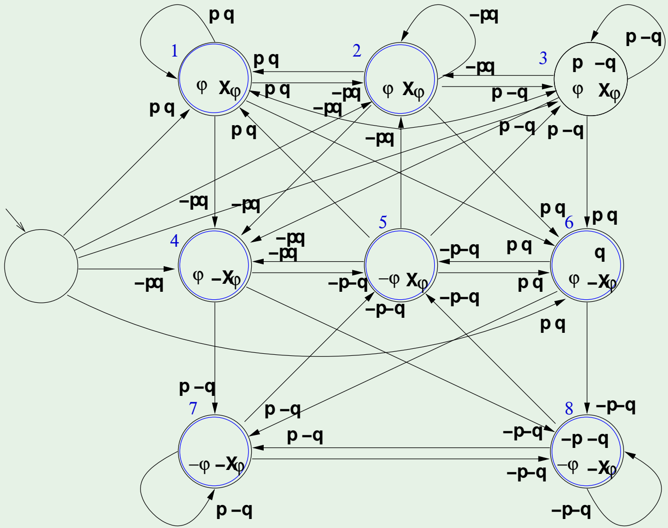
$$F_{T_\psi} := \{sat(\neg(\varphi_1 \mathbf{U} \varphi_2) \vee \varphi_2)) \ s.t.\ (\varphi_1 \mathbf{U} \varphi_2) \ occurs\ [positively] in\ \psi\}$$

Example: $\psi := p\mathbf{U}q$ [cont.]



Note: easily transformed into a generalized Büchi automaton

Example: $\psi := p\mathbf{U}q$ [cont.]



Note: easily transformed into a generalized Büchi automaton

Symbolic Representation of T_ψ

- State variables: one Boolean variable for each formula in $el(\psi)$
  - EX: p, q and x and primed versions p', q' and x'
    [x is a Boolean label for $\mathbf{X}(p\mathbf{U}q)$]
- $sat(\varphi_i)$:
  - $sat(p) := p$, s.t. p Boolean state variable
  - $sat(\neg\varphi_1) := \neg sat(\varphi_1)$
  - $sat(\varphi_1 \wedge \varphi_2) := sat(\varphi_1) \wedge sat(\varphi_2)$
  - $sat(\mathbf{X}\varphi_i) := x_{[\mathbf{X}\varphi_i]}$, s.t. $x_{[\mathbf{X}\varphi_i]}$ Boolean state variable
  - $sat(\varphi_1\mathbf{U}\varphi_2) := sat(\varphi_2) \vee (sat(\varphi_1) \wedge sat(\mathbf{X}(\varphi_1\mathbf{U}\varphi_2)))$

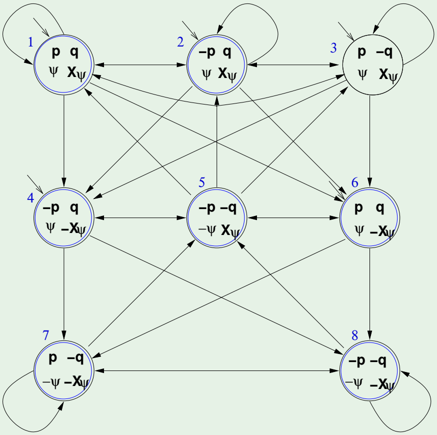
  $\Longrightarrow$ $sat(\varphi_1\mathbf{U}\varphi_2) := sat(\varphi_2) \vee (sat(\varphi_1) \wedge x_{[\mathbf{X}\varphi_1\mathbf{U}\varphi_2]})$
- ...

Symbolic Representation of T_ψ [cont.]

- ...
- Initial states: $I_{T_\psi} = sat(\psi)$
  - EX: $I(p, q, x) = q \vee (p \wedge x)$
- Transition Relation: $R_{T_\psi}(s, s') = \bigcap_{\mathbf{X}\varphi_i \in el(\psi)} \{(s, s') \mid s \in sat(\mathbf{X}\varphi_i) \Leftrightarrow s' \in sat(\varphi_i)\}$
  - $R_{T_\psi} = \bigwedge_{\mathbf{X}\varphi_i \in el(\psi)} (sat(\mathbf{X}\varphi_i) \leftrightarrow sat'(\varphi_i))$
    where $sat'(\varphi_i)$ is $sat(\varphi_i)$ on primed variables
  - EX: $R_{T_\psi}(p, q, x, p', q', x') = x \leftrightarrow (q' \vee (p' \wedge x'))$
- Fairness Conditions: $F_{T_\psi} := \{sat(\neg(\varphi_1 \mathbf{U} \varphi_2) \vee \varphi_2))\ s.t.\ (\varphi_1 \mathbf{U} \varphi_2)\ occurs\ [positively]\ in\ \psi\}$
  - EX: $F_{T_\psi}(p, q, x) = \neg(q \vee (p \wedge x)) \vee q = \ldots = \neg p \vee \neg x \vee q$

Symbolic Representation of T_ψ: Examples

- $I_{T_\psi}(p,q,x) = q \vee (p \wedge x)$
  - $1: \quad \{p,q,x\} \models I_{T_\psi}$
  - $3: \quad \{p,\neg q,x\} \models I_{T_\psi}$
  - $\cancel{5}: \quad \{\neg p,\neg q,x\} \not\models I_{T_\psi}$
- $R_{T_\psi}(p,q,x,p',q',x') = \qquad x \leftrightarrow (q' \vee (p' \wedge x'))$
  - $1 \Rightarrow 1: \quad \{p,q,x,p',q',x'\} \models R_{T_\psi}$
  - $6 \Rightarrow 7: \quad \{p,q,\neg x,p',\neg q',\neg x'\} \models R_{T_\psi}$
  - $6 \not\Rightarrow 1: \quad \{p,q,\neg x,p',q',x'\} \not\models R_{T_\psi}$
- $F_{T_\psi}(p,q,x) = \neg p \vee \neg x \vee q$
  - $1: \quad \{p,q,x\} \models F_{T_\psi}$
  - $5: \quad \{\neg p,\neg q,x\} \models F_{T_\psi}$
  - $\cancel{3}: \quad \{p,\neg q,x\} \not\models F_{T_\psi}$



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  $6 \Rightarrow 7: \quad \{p,q,\neg x,p',\neg q',\neg x'\} \models R_{T_\psi}$

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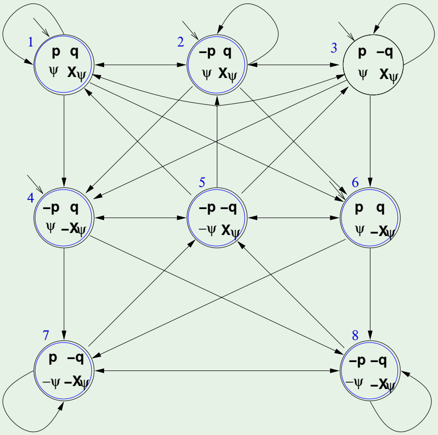

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  $\not 5: \quad \{\neg p, \neg q, x\} \not\models I_{T_\psi}$

- $R_{T_\psi}(p, q, x, p', q', x') = \quad x \leftrightarrow (q' \vee (p' \wedge x'))$

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  $6 \Rightarrow 7: \quad \{p, q, \neg x, p', \neg q', \neg x'\} \models R_{T_\psi}$

  $6 \not\Rightarrow 1: \quad \{p, q, \neg x, p', q', x'\} \not\models R_{T_\psi}$

- $F_{T_\psi}(p, q, x) = \neg p \vee \neg x \vee q$

  $1: \quad \{p, q, x\} \models F_{T_\psi}$

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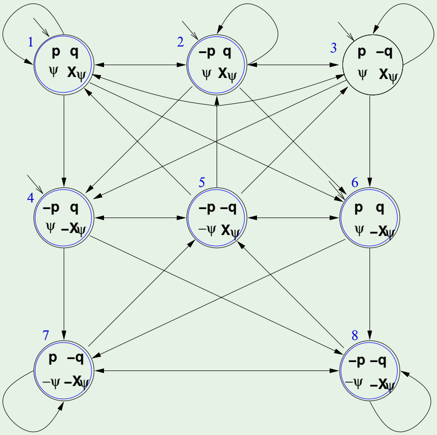
  $\not 3: \quad \{p, \neg q, x\} \not\models F_{T_\psi}$



Symbolic Representation of T_ψ: Examples

- $I_{T_\psi}(p, q, x) = q \vee (p \wedge x)$

  $1: \quad \{p, q, x\} \models I_{T_\psi}$

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  $\cancel{5}: \quad \{\neg p, \neg q, x\} \not\models I_{T_\psi}$

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  $6 \Rightarrow 7: \quad \{p, q, \neg x, p', \neg q', \neg x'\} \models R_{T_\psi}$

  $6 \not\Rightarrow 1: \quad \{p, q, \neg x, p', q', x'\} \not\models R_{T_\psi}$

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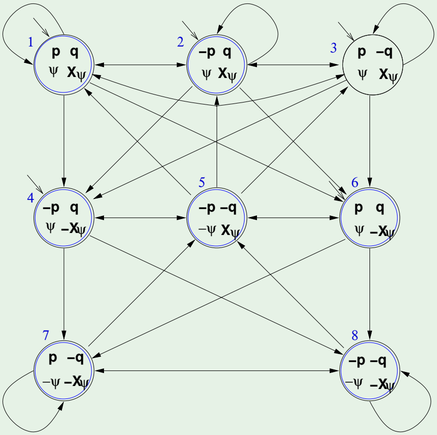
  $\cancel{3}: \quad \{p, \neg q, x\} \not\models F_{T_\psi}$



Outline

Computing the product $P := T_\psi \times M$

- Given $M := \langle S_M, I_M, R_M, L_M \rangle$ and $T_\psi := \langle S_{T_\psi}, I_{T_\psi}, R_{T_\psi}, L_{T_\psi}, F_{T_\psi} \rangle$, we compute the product $P := T_\psi \times M = \langle S, I, R, L, F \rangle$ as follows:
  - $S := \{(s, s') \mid s \in S_{T_\psi},\ s' \in S_M \text{ and } L_M(s')|_\psi = L_{T_\psi}(s)\}$
  - $I := \{(s, s') \mid s \in I_{T_\psi},\ s' \in I_M \text{ and } L_M(s')|_\psi = L_{T_\psi}(s)\}$
  - Given $(s, s'), (t, t') \in S$, $((s, s'), (t, t')) \in R$ iff $(s, t) \in R_{T_\psi}$ and $(s', t') \in R_M$
  - $L((s, s')) = L_{T_\psi}(s) \cup L_M(s')$
- Extension of $sat()$ and F_{T_ψ} to P:
  $(s, s') \in sat(\psi) \Longleftrightarrow s \in sat(\psi)$
  $F := \{sat(\neg(\varphi_1 \mathbf{U} \varphi_2) \vee \varphi_2) \ s.t.\ (\varphi_1 \mathbf{U} \varphi_2) \ occurs\ [positively] in\ \psi\}$

Computing the product $P := T_\psi \times M$ symbolically

Let V, W be the array of Boolean state variables of T_ψ and M respectively:

- Initial states: $I(V \cup W) = I_{T_\psi}(V) \wedge I_M(W)$
- Transition Relation: $R(V \cup W, V' \cup W') = R_{T_\psi}(V, V') \wedge R_M(W, W')$
- Fairness conditions: $\{F_1(V \cup W), ..., F_k(V \cup W)\} = \{F_{T_\psi 1}(V), ..., F_{T_\psi k}(V)\}$

Outline

Main theorem [Clarke, Grumberg & Hamaguchi; 94]

Theorem

THEOREM: $M.s' \models \mathbf{E}\psi$ iff there is a state s in T_ψ s.t. $(s, s') \in sat(\psi)$ and $T_\psi \times M, (s, s') \models \mathbf{EG}true$ under the fairness conditions:

$$\{sat(\neg(\varphi_1 \mathbf{U} \varphi_2) \vee \varphi_2))\ s.t.\ (\varphi_1 \mathbf{U} \varphi_2)\ occurs\ in\ \psi\}.$$

$\Longrightarrow$ $M \models \mathbf{E}\psi$ iff $T_\psi \times M \models \mathbf{E}_f\mathbf{G}true$

$\Longrightarrow$ $M \models \neg\psi$ iff $T_\psi \times M \not\models \mathbf{E}_f\mathbf{G}true$

- LTL M.C. reduced to Fair CTL M.C.!!!
- Symbolic OBDD-based techniques apply.

Note

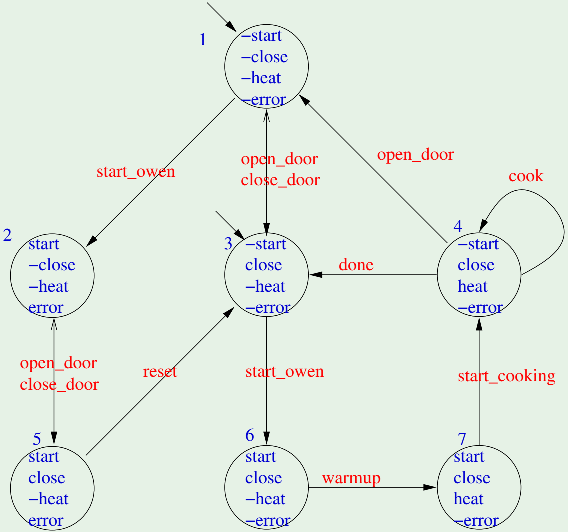
The transition relation R of $T_\psi \times M$ may not be total.
$\Longrightarrow$ `Check_FairEG` does not consider states without successors, restricting R to the remaining states.

Outline

A microwave oven

- 4 state variables: start, close, heat, error
- Actions (implicit): start_oven,open_door, close_door, reset, warmup, start_cooking, cook, done
- Error situation: if oven is started while the door is open
- Represented as a Kripke structure (and hence as a OBDD's)

A microwave oven [cont.]



A microwave oven: symbolic representation

- Initial states: $I_M(s, c, h, e) = \neg s \wedge \neg h \wedge \neg e$
- Transition relation: $R_M(s, c, h, e, s', c', h', e') =$ [a simplification of]

$$
\begin{array}{lllllllllll}
(& \neg s \wedge \neg c \wedge \neg h \wedge \neg e \wedge \neg s' \wedge c' \wedge \neg h' \wedge \neg e') & \vee & (\mathit{close\_door}, \ \mathit{no\ error}) \\
(& s \wedge \neg c \wedge \neg h \wedge e \wedge s' \wedge c' \wedge \neg h' \wedge e') & \vee & (\mathit{close\_door}, \ \mathit{error}) \\
(& \neg s \wedge c \wedge \neg e \wedge \neg s' \wedge \neg c' \wedge \neg h' \wedge \neg e') & \vee & (\mathit{open\_door}, \ \mathit{no\ error}) \\
(& s \wedge c \wedge \neg h \wedge e \wedge s' \wedge \neg c' \wedge \neg h' \wedge e') & \vee & (\mathit{open\_door}, \ \mathit{error}) \\
(& \neg s \wedge c \wedge \neg h \wedge \neg e \wedge s' \wedge c' \wedge \neg h' \wedge \neg e') & \vee & (\mathit{start\_oven}, \ \mathit{no\ error}) \\
(& \neg s \wedge \neg c \wedge \neg h \wedge \neg e \wedge s' \wedge \neg c' \wedge \neg h' \wedge e') & \vee & (\mathit{start\_oven}, \ \mathit{error}) \\
(& s \wedge c \wedge \neg h \wedge e \wedge \neg s' \wedge c' \wedge \neg h' \wedge \neg e') & \vee & (\mathit{reset}) \\
(& s \wedge c \wedge \neg h \wedge \neg e \wedge s' \wedge c' \wedge h' \wedge \neg e') & \vee & (\mathit{warmup}) \\
(& s \wedge c \wedge h \wedge \neg e \wedge \neg s' \wedge c' \wedge h' \wedge \neg e') & \vee & (\mathit{start\_cooking}) \\
(& \neg s \wedge c \wedge h \wedge \neg e \wedge \neg s' \wedge c' \wedge h' \wedge \neg e') & \vee & (\mathit{cook}) \\
(& \neg s \wedge c \wedge h \wedge \neg e \wedge \neg s' \wedge c' \wedge \neg h' \wedge \neg e') & & (\mathit{done})
\end{array}
$$

Note: the third row represents two transitions: $3 \rightarrow 1$ and $4 \rightarrow 1$.

LTL Specification

- “necessarily, the oven’s door eventually closes and, till there, the oven does not heat”:

$$M \models \neg heat \ \mathbf{U} \ close,$$

i.e.,

$$M \models \neg\mathbf{E}\neg(\neg heat \ \mathbf{U} \ close)$$

Tableau construction for $\psi = \neg(\neg \textit{heat}\ \mathbf{U}\ \textit{close})$

- $\varphi := \neg\psi = (\neg \textit{heat}\ \mathbf{U}\ \textit{close})$
- Tableaux expansion: $\psi = \neg(\neg \textit{heat}\ \mathbf{U}\ \textit{close}) = \neg(\textit{close} \vee (\neg \textit{heat} \wedge \mathbf{X}(\neg \textit{heat}\ \mathbf{U}\ \textit{close})))$
- $\textit{el}(\psi) = \textit{el}(\varphi) = \{\textit{heat}, \textit{close}, \mathbf{X}\varphi\}$ $(\{h, c, \mathbf{X}\varphi\})$
- States:

$$1 := \{\neg h, c, \mathbf{X}\varphi\},\ 2 := \{h, c, \mathbf{X}\varphi\},\ 3 := \{\neg h, \neg c, \mathbf{X}\varphi\},$$
$$4 := \{h, c, \neg\mathbf{X}\varphi\},\ 5 := \{h, \neg c, \mathbf{X}\varphi\},\ 6 := \{\neg h, c, \neg\mathbf{X}\varphi\},$$
$$7 := \{\neg h, \neg c, \neg\mathbf{X}\varphi\},\ 8 := \{h, \neg c, \neg\mathbf{X}\varphi\}$$

Tableau construction for $\psi = \neg(\neg$*heat* **U** *close*$)$ [cont.]

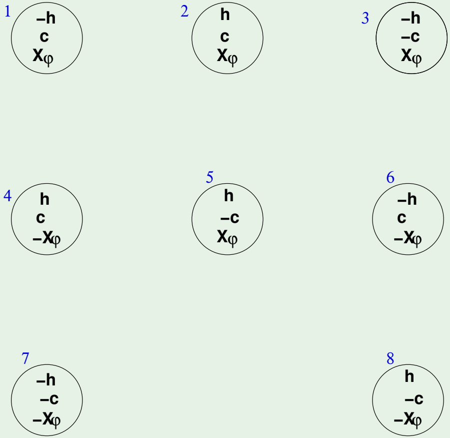


Tableau construction for $\psi = \neg(\neg heat\ \mathbf{U}\ close)$

- ...
- States:

$$1 := \{\neg h, c, \mathbf{X}\varphi\},\ 2 := \{h, c, \mathbf{X}\varphi\},\ 3 := \{\neg h, \neg c, \mathbf{X}\varphi\},$$
$$4 := \{h, c, \neg\mathbf{X}\varphi\},\ 5 := \{h, \neg c, \mathbf{X}\varphi\},\ 6 := \{\neg h, c, \neg\mathbf{X}\varphi\},$$
$$7 := \{\neg h, \neg c, \neg\mathbf{X}\varphi\},\ 8 := \{h, \neg c, \neg\mathbf{X}\varphi\}$$

- *sat*():

$$sat(h) = \{2,4,5,8\} \implies sat(\neg h) = \{1,3,6,7\},$$
$$sat(c) = \{1,2,4,6\} \implies sat(\neg c) = \{3,5,7,8\},$$
$$sat(\mathbf{X}\varphi) = \{1,2,3,5\} \implies sat(\neg\mathbf{X}\varphi) = \{4,6,7,8\},$$
$$sat(\varphi) = sat(c) \cup (sat(\neg h) \cap sat(\mathbf{X}(\neg h\ \mathbf{U}\ c))) = \{1,2,3,4,6\}$$
$$\implies sat(\psi) = sat(\neg\varphi) = \{5,7,8\}$$

Tableau construction for $\psi = \neg(\neg\textit{heat}\ \mathbf{U}\ \textit{close})$ [cont.]

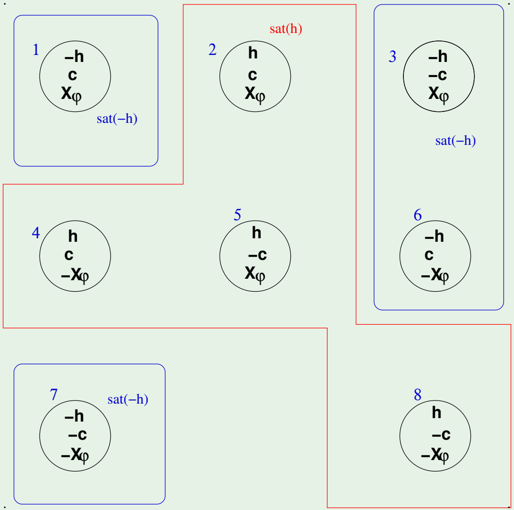


Tableau construction for $\psi = \neg(\neg \textit{heat}\ \mathbf{U}\ \textit{close})$ [cont.]

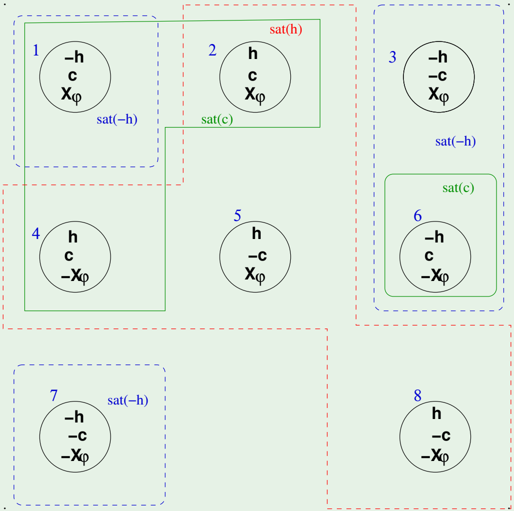


Tableau construction for $\psi = \neg(\neg heat \ \mathbf{U} \ close)$ [cont.]

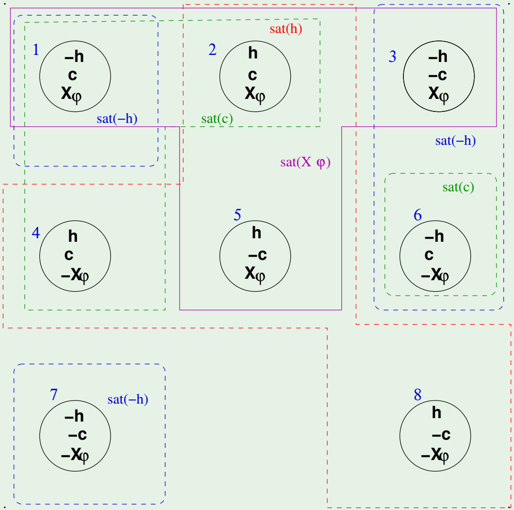


Tableau construction for $\psi = \neg(\neg heat \ \mathbf{U}\ close)$ [cont.]

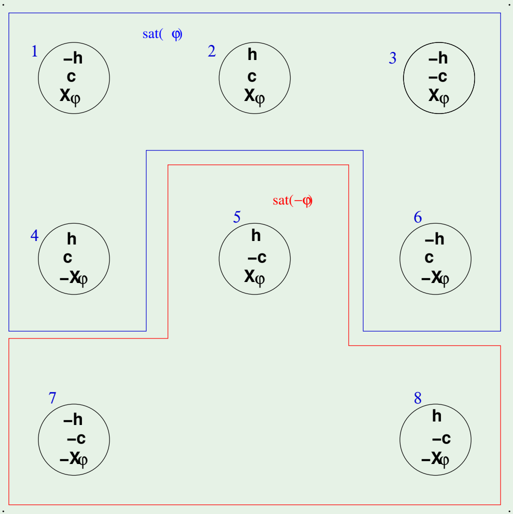


Tableau construction for $\psi = \neg(\neg \textit{heat}\ \mathbf{U}\ \textit{close})$ [cont.]

- ...
- $\textit{sat}()$:

$$\begin{array}{l} \textit{sat}(h) = \{2,4,5,8\} \implies \textit{sat}(\neg h) = \{1,3,6,7\}, \\ \textit{sat}(c) = \{1,2,4,6\} \implies \textit{sat}(\neg c) = \{3,5,7,8\}, \\ \textit{sat}(\mathbf{X}\varphi) = \{1,2,3,5\} \implies \textit{sat}(\neg \mathbf{X}\varphi) = \{4,6,7,8\}, \\ \textit{sat}(\varphi) = \textit{sat}(c) \cup (\textit{sat}(\neg h) \cap \textit{sat}(\mathbf{X}(\neg h\ \mathbf{U}\ c))) = \{1,2,3,4,6\} \end{array}$$

- Initial states I: $\textit{sat}(\psi) = \textit{sat}(\neg\varphi) = \{5,7,8\}$
- Transition Relation R:
  - add an edge from every state in $\textit{sat}(\mathbf{X}\varphi)$ to every state in $\textit{sat}(\varphi)$
  - add an edge from every state in $\textit{sat}(\neg\mathbf{X}\varphi)$ to every state in $\textit{sat}(\neg\varphi)$

Tableau construction for $\psi = \neg(\neg heat\ \mathbf{U}\ close)$ [cont.]

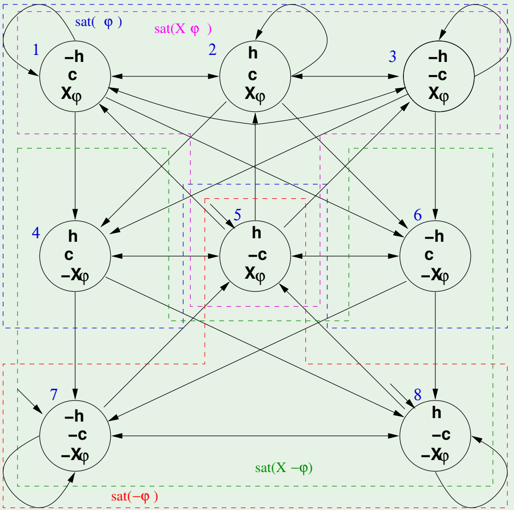
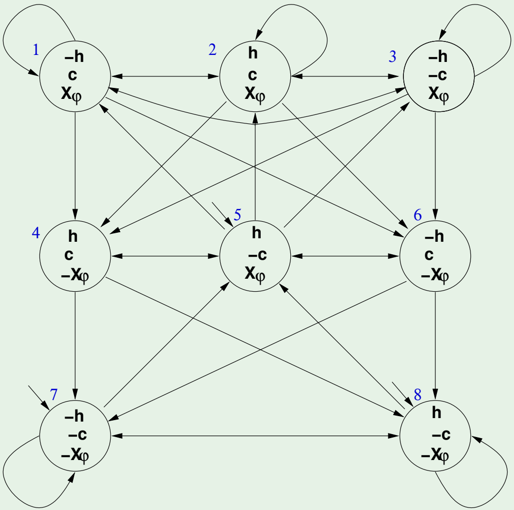


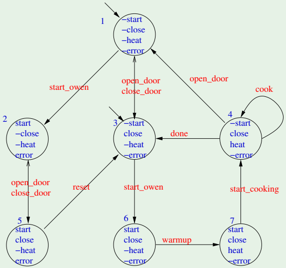
Tableau construction for $\psi = \neg(\neg heat \ \mathbf{U}\ close)$ [cont.]

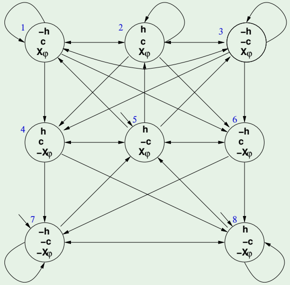


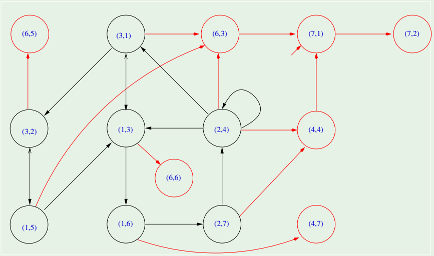
Symbolic representation of T_ψ, s.t. $\psi := \neg(\neg h \mathbf{U} c)$

- State variables: h, c and x and primed versions h', c' and x'
  [x is a Boolean label for $\mathbf{X}(\neg h \mathbf{U} c)$]
- Initial states: $I_{T_\psi} = sat(\psi)$
  $\Longrightarrow I(h, c, x) = \neg(c \vee (\neg h \wedge x))$
- Transition Relation: $R_{T_\psi} = \bigwedge_{\mathbf{X}\varphi_i \in el(\psi)} (sat(\mathbf{X}\varphi_i) \leftrightarrow sat'(\varphi_i))$
  $\Longrightarrow R_{T_\psi}(h, c, x, h', c', x') = x \leftrightarrow (c' \vee (\neg h' \wedge x'))$
- Fairness Property: (due to negative polarity of $(\neg h\, \mathbf{U} c)$ in ψ):
  $F_{T_\psi}(h, c, x) = \top$

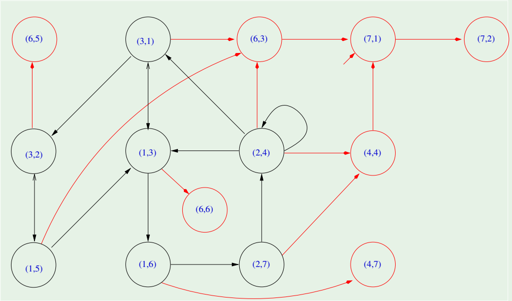
Product $P = T_\psi \times M$







Product $P = T_\psi \times M$ [cont.]



- $P = T_\psi \times M$ (reachable states only)
- compute [**EG**_true_] (e.g. by Emerson-Lei):

  $\Longrightarrow$ states $(4,4)$, $(4,7)$, $(6,3)$, $(6,5)$, $(6,6)$, $(7,1)$, $(7,2)$ are not part of a (fair) infinite path

  $\Longrightarrow$ no initial states in [**EG**_true_] ((7.1) has been removed).

  $\Longrightarrow$ $T_\psi \times M \not\models$ **EG**_true_

  $\Longrightarrow$ Property verified!
- N.B.: fairness condition $\top$ irrelevent here

Product $P = T_\psi \times M$: symbolic representation

- Initial states: $I(s,c,h,e,x) = (\neg s \wedge \neg h \wedge \neg e) \wedge \neg(c \vee (\neg h \wedge x)) = \neg s \wedge \neg h \wedge \neg e \wedge \neg c \wedge \neg x$
- Transition relation: $R(s,c,h,e,x,s',c',h',e',x') =$ (an OBDD for)

$$
\begin{array}{ll}
\multicolumn{2}{l}{(x \leftrightarrow (c' \vee (\neg h' \wedge x'))) \wedge (} \\
(\ \neg s \wedge \neg c \wedge \neg h \wedge \neg e \wedge \neg s' \wedge c' \wedge \neg h' \wedge \neg e') \vee & (close\_door,\ no\ error) \\
(\ s \wedge \neg c \wedge \neg h \wedge e \wedge s' \wedge c' \wedge \neg h' \wedge e') \vee & (close\_door,\ error) \\
(\ \neg s \wedge c \wedge \neg e \wedge \neg s' \wedge \neg c' \wedge \neg h' \wedge \neg e') \vee & (open\_door,\ no\ error) \\
(\ s \wedge c \wedge \neg h \wedge e \wedge s' \wedge \neg c' \wedge \neg h' \wedge e') \vee & (open\_door,\ error) \\
(\ \neg s \wedge c \wedge \neg h \wedge \neg e \wedge s' \wedge c' \wedge \neg h' \wedge \neg e') \vee & (start\_oven,\ no\ error) \\
(\ \neg s \wedge \neg c \wedge \neg h \wedge \neg e \wedge s' \wedge \neg c' \wedge \neg h' \wedge e') \vee & (start\_oven,\ error) \\
(\ s \wedge c \wedge \neg h \wedge e \wedge \neg s' \wedge c' \wedge \neg h' \wedge \neg e') \vee & (reset) \\
(\ s \wedge c \wedge \neg h \wedge \neg e \wedge s' \wedge c' \wedge h' \wedge \neg e') \vee & (warmup) \\
(\ s \wedge c \wedge h \wedge \neg e \wedge \neg s' \wedge c' \wedge h' \wedge \neg e') \vee & (start\_cooking) \\
(\ \neg s \wedge c \wedge h \wedge \neg e \wedge \neg s' \wedge c' \wedge h' \wedge \neg e') \vee & (cook) \\
(\ \neg s \wedge c \wedge h \wedge \neg e \wedge \neg s' \wedge c' \wedge \neg h' \wedge \neg e') & (done) \\
)
\end{array}
$$

[**EG***true*]: symbolic representation

- Emerson-Lei returns (an OBDD equivalent to):

$$
\begin{array}{lr}
\textbf{EG}\mathit{true} = & \\
(\ \neg s \wedge \neg c \wedge \neg h \wedge \neg e \wedge\ x) \vee & (3,1) \\
(\ \ s \wedge \neg c \wedge \neg h \wedge\ e \wedge\ x) \vee & (3,2) \\
(\ \neg s \wedge\ c \wedge \neg h \wedge \neg e \wedge\ x) \vee & (1,3) \\
(\ \neg s \wedge\ c \wedge\ h \wedge \neg e \wedge\ x) \vee & (2,4) \\
(\ \ s \wedge\ c \wedge \neg h \wedge\ e \wedge\ x) \vee & (1,5) \\
(\ \ s \wedge\ c \wedge \neg h \wedge \neg e \wedge\ x) \vee & (1,5) \\
(\ \ s \wedge\ c \wedge\ h \wedge \neg e \wedge\ x) \vee & (2,7) \\
\quad \ldots & \text{(other unreachables states)}
\end{array}
$$

- Initial states: $I(s,c,h,e,x) = \neg s \wedge \neg h \wedge \neg e \wedge \neg c \wedge \neg x$

$\Longrightarrow$ $I(s,c,h,e,x) \not\models \textbf{EG}\mathit{true}$

$\Longrightarrow$ $I \not\subseteq [\textbf{EG}\mathit{true}]$

$\Longrightarrow$ $T_\psi \times M \not\models \textbf{EG}\mathit{true}$

$\Longrightarrow$ Property verified!

The property verified is...

Outline

Ex: Symbolic CTL Model Checking

Given the following finite state machine expressed in NuSMV input language:

```
MODULE main
VAR v1 : boolean; v2 : boolean;
INIT (!v1 & !v2)
TRANS (next(v1) <-> !v1) & (next(v2) <-> (v1<->v2))
```

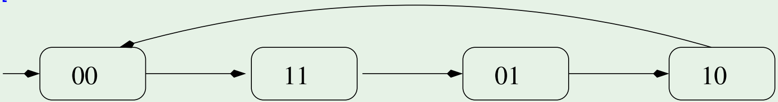
and consider the property $P \stackrel{\text{def}}{=} (v_1 \wedge v_2)$. Write:

- the Boolean formulas $I(v_1, v_2)$ and $T(v_1, v_2, v_1', v_2')$ representing respectively the initial states and the transition relation of M.

  [Solution: $I(v_1, v_2)$ is $(\neg v_1 \wedge \neg v_2)$, $T(v_1, v_2, v_1', v_2')$ is $(v_1' \leftrightarrow \neg v_1) \wedge (v_2' \leftrightarrow (v_1 \leftrightarrow v_2))$]

- the graph representing the FSM. (Assume the notation "$v_1 v_2$" for labeling the states: e.g. "10" means "$v_1 = 1, v_2 = 0$".)

  [Solution:



]

Ex: Symbolic CTL Model Checking (cont.)

- the Boolean formula representing symbolically **EX**P. [The formula must be computed symbolically, not simply inferred from the graph of the previous question!]

  [Solution:

$$
\begin{array}{rcl}
\mathbf{EX}(P) & = & \exists v_1', v_2'.(T(v_1, v_2, v_1', v_2') \wedge P(v_1', v_2')) \\
 & = & \exists v_1', v_2'.((v_1' \leftrightarrow \neg v_1) \wedge (v_2' \leftrightarrow (v_1 \leftrightarrow v_2)) \wedge \underbrace{(v_1' \wedge v_2')}_{\Longrightarrow v_1'=\top, v_2'=\top}) \\
 & = & \overbrace{(\neg v_1 \wedge \neg v_2)}^{v_1'=\top, v_2'=\top} \vee \bot \vee \bot \vee \bot \\
 & = & (\neg v_1 \wedge \neg v_2)
\end{array}
$$

  .]

Ex: Symbolic CTL Model Checking

Given the following finite state machine expressed in NuSMV input language:

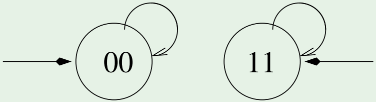
```
VAR     v1 : boolean;  v2 : boolean;
INIT    init(v1) <-> init(v2)
TRANS   (v1 <-> next(v2)) &   (v2 <-> next(v1));
```

write:

- the Boolean formulas $I(v_1, v_2)$ and $T(v_1, v_2, v_1', v_2')$ representing the initial states and the transition relation of M respectively.

  [Solution: $I(v_1, v_2)$ is $(v_1 \leftrightarrow v_2)$, $T(v_1, v_2, v_1', v_2')$ is $(v_1 \leftrightarrow v_2') \wedge (v_2 \leftrightarrow v_1')$]

- the graph representing the FSM. (Assume the notation "$v_1 v_2$" for labeling the states. E.g., "10" means "$v_1 = 1, v_2 = 0$".)

  [Solution: 

]

Ex: Symbolic CTL Model Checking (cont.)

- the Boolean formula $R^1(v'_1, v'_2)$ representing the set of states which can be reached after exactly 1 step. NOTE: this must be computed symbolically, not simply deduced from the graph of question b).

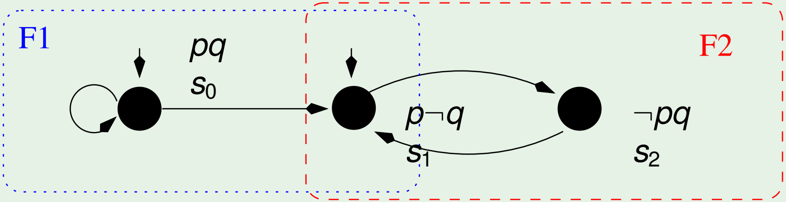
  [Solution:

$$
\begin{array}{rcl}
R^1(v'_1, v'_2) & = & \exists v_1, v_2.(I(v_1, v_2) \wedge T(v_1, v_2, v'_1, v'_2)) \\
 & = & \exists v_1, v_2.((v_1 \leftrightarrow v_2) \wedge (v_1 \leftrightarrow v'_2) \wedge (v_2 \leftrightarrow v'_1)) \\
 & = & ((v_1 \leftrightarrow v_2) \wedge (v_1 \leftrightarrow v'_2) \wedge (v_2 \leftrightarrow v'_1))[v_1 = \bot, v_2 = \bot] \vee \\
 & & ((v_1 \leftrightarrow v_2) \wedge (v_1 \leftrightarrow v'_2) \wedge (v_2 \leftrightarrow v'_1))[v_1 = \bot, v_2 = \top] \vee \\
 & & ((v_1 \leftrightarrow v_2) \wedge (v_1 \leftrightarrow v'_2) \wedge (v_2 \leftrightarrow v'_1))[v_1 = \top, v_2 = \bot] \vee \\
 & & ((v_1 \leftrightarrow v_2) \wedge (v_1 \leftrightarrow v'_2) \wedge (v_2 \leftrightarrow v'_1))[v_1 = \top, v_2 = \top] \\
 & = & (\neg v'_1 \wedge \neg v'_2) \vee \bot \vee \bot \vee (v'_1 \wedge v'_2) \\
 & = & (\neg v'_1 \wedge \neg v'_2) \vee (v'_1 \wedge v'_2) \\
 & = & (v'_1 \leftrightarrow v'_2)
\end{array}
$$

  .]

Ex: Fair CTL Model Checking

Consider the following *fair* Kripke Model M:



For each of the following facts, say if it is true or false in CTL.

(a) $M \models \mathbf{AF}\neg p$
[Solution: true]

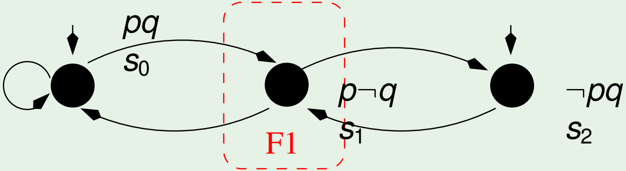
(b) $M \models \mathbf{A}(p\mathbf{U}\neg q)$
[Solution: true]

(c) $M \models \mathbf{AX}\neg q$
[Solution: false]

(d) $M \models \mathbf{AGAF}\neg p$
[Solution: true]

Ex: Fair CTL Model Checking

Consider the following *fair* Kripke Model M:



For each of the following facts, say if it is true or false in CTL.

(a) $M \models \mathbf{EF}(p \land q)$
[Solution: true]

(b) $M \models \mathbf{AGAF}p$
[Solution: true]

(c) $M \models \mathbf{AF}\neg q$
[Solution: true]

(d) $M \models \mathbf{AG}(\neg p \lor \neg q)$
[Solution: false]

Ex: Symbolic LTL Model Checking

Given the following LTL formula: $\varphi \stackrel{\text{def}}{=} \neg((\mathbf{GF}p \wedge \mathbf{GF}q) \to \mathbf{GF}r)$

(a) Compute the Negative Normal Form of φ ($NNF(\varphi)$).

[Solution:
$$\begin{array}{rcl} \varphi & \Longleftrightarrow & \neg((\mathbf{GF}p \wedge \mathbf{GF}q) \to \mathbf{GF}r) \\ & \Longleftrightarrow & \neg(\neg(\mathbf{GF}p \wedge \mathbf{GF}q) \vee \mathbf{GF}r) \\ & \Longleftrightarrow & (\mathbf{GF}p \wedge \mathbf{GF}q \wedge \neg\mathbf{GF}r) \\ & \Longleftrightarrow & (\mathbf{GF}p \wedge \mathbf{GF}q \wedge \mathbf{FG}\neg r) \Longleftrightarrow NNF(\varphi) \end{array}$$
]

(b) Compute the set of elementary subformulas of φ.

[Solution: First write the formula in terms of $\mathbf{X}$ and $\mathbf{U}$'s (write "$\mathbf{F}\psi$" for "$\top\mathbf{U}\psi$"):

$$\begin{array}{rcl} \varphi & \Longleftrightarrow & \neg((\mathbf{GF}p \wedge \mathbf{GF}q) \to \mathbf{GF}r) \\ & \Longleftrightarrow & \neg((\neg\mathbf{F}\neg\mathbf{F}p \wedge \neg\mathbf{F}\neg\mathbf{F}q) \to \neg\mathbf{F}\neg\mathbf{F}r) \end{array}$$

$el(\mathbf{F}\neg\mathbf{F}p) = \{\mathbf{XF}\neg\mathbf{F}p\} \cup el(\neg\mathbf{F}p) = \{\mathbf{XF}\neg\mathbf{F}p\} \cup \{\mathbf{XF}p\} \cup el(p) = \{\mathbf{XF}\neg\mathbf{F}p, \mathbf{XF}p, p\}$.

$$\begin{array}{rcl} \text{Hence: } el(\varphi) & = & el(\neg((\neg\mathbf{F}\neg\mathbf{F}p \wedge \neg\mathbf{F}\neg\mathbf{F}q) \to \neg\mathbf{F}\neg\mathbf{F}r)) \\ & = & el(\mathbf{F}\neg\mathbf{F}p) \cup el(\mathbf{F}\neg\mathbf{F}q) \cup el(\mathbf{F}\neg\mathbf{F}r) \\ & = & \{\mathbf{XF}\neg\mathbf{F}p, \mathbf{XF}p, p, \mathbf{XF}\neg\mathbf{F}q, \mathbf{XF}q, q, \mathbf{XF}\neg\mathbf{F}r, \mathbf{XF}r, r\} \end{array}$$
]

(c) What is the (maximum) number of states of a fair Kripke Model representing φ?

[Solution: By definition it is $2^{|el(\varphi)|} = 2^9 = 512$.]

Ex: Symbolic LTL Model Checking

Given the following LTL formula $\psi \stackrel{\text{def}}{=} \neg\mathbf{F}\neg p$, compute and draw the tableau $\mathcal{T}_\psi$ of ψ. [Solution:

(i) The set of elementary subformulas of ψ is $el(\psi) \stackrel{\text{def}}{=} \{p, \mathbf{XF}\neg p\}$. Hence, the set of states is

$$\{s_1 : (p, \neg\mathbf{XF}\neg p),\ s_2 : (p, \mathbf{XF}\neg p),\ s_3 : (\neg p, \neg\mathbf{XF}\neg p),\ s_4 : (\neg p, \mathbf{XF}\neg p)\}$$

(ii) The set of initial states of $\mathcal{T}_\psi$ is $sat(\psi) \stackrel{\text{def}}{=} S \setminus (sat(\neg p) \cup sat(\mathbf{XF}\neg p)) = \{s_1\}$.

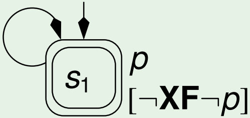
(iii) Since s_1 is the only state in $sat(\neg\mathbf{F}\neg p)$, then s_1 is the only successor of itself, so that the only relevant transition is a self-loop over s_1.
(One can also —un-necessarily— draw all transitions from states where $\neg\mathbf{XF}\neg p$ holds into $\{s_1\}$ and from from states where $\mathbf{XF}\neg p$ holds into $\{s_2, s_3, s_4\}$.)

(iv) There is one $\mathbf{U}$-subformula, $\mathbf{F}\neg p$, so that there is one fairness condition defined as $sat(\neg\mathbf{F}\neg p \vee \neg p)$. Since $\mathbf{F}\neg p$ is false in s_1, then s_1 is part of the fairness condition. [Alternatively: there is no positive $\mathbf{U}$-subformula, so that we must add a $\mathbf{AGAF}\top$ fairness condition, which is equivalent to say that all states belong to the fairness condition.]

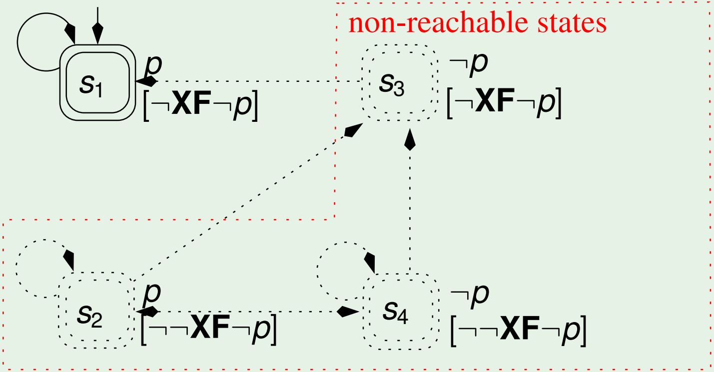
]

Ex: Symbolic LTL Model Checking (cont.)

[Solution:



or, alternatively without simplifications:



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Ex: Symbolic LTL Model Checking

Given the following LTL formula $\psi \stackrel{\text{def}}{=} \mathbf{G}p$, compute and draw the tableau $\mathcal{T}_\psi$ of ψ. [Without converting anything into $\mathbf{X}, \mathbf{U}$].
[Solution:

(i) The set of elementary subformulas of ψ is $el(\psi) \stackrel{\text{def}}{=} \{p, \mathbf{XG}p\}$. Hence, the set of states is

$$\{s_1 : (p, \mathbf{XG}p),\ s_2 : (p, \neg\mathbf{XG}p),\ s_3 : (\neg p, \mathbf{XG}p),\ s_4 : (\neg p, \neg\mathbf{XG}p)\}$$

(ii) The set of initial states of $\mathcal{T}_\psi$ is $sat(\psi) \stackrel{\text{def}}{=} sat(p) \cap sat(\mathbf{XG}p) = \{s_1\}$.

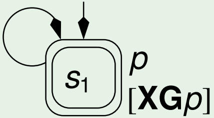
(iii) Since s_1 is the only state in $sat(\mathbf{G}p)$, then s_1 is the only successor of itself, so that the only relevant transition is a self-loop over s_1.
(One can also —un-necessarily— draw all transitions from states where $\mathbf{XG}p$ holds into $\{s_1\}$ and from from states where $\neg\mathbf{XG}p$ holds into $\{s_2, s_3, s_4\}$.)

(iv) Since there is no "**U**" subformula, we must add a $\mathbf{AGAF}\top$ fairness condition, which is equivalent to say that all states belong to the fairness condition.

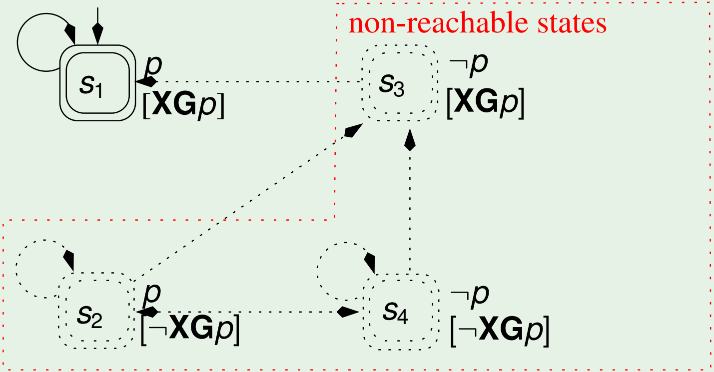
]

Ex: Symbolic LTL Model Checking (cont.)

[Solution:



or, alternatively without simplifications:



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